i.e.,
$\left(\begin{array}{cccc}D_{T} & 0 & \cdots & 0 \\ 0 & D_{T} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{T}\end{array}\right) y=\left(\begin{array}{cccc}D_{T} & 0 & \cdots & 0 \\ 0 & D_{T} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{T}\end{array}\right) X \beta+\left(\begin{array}{cccc}D_{T} & 0 & \cdots & 0 \\ 0 & D_{T} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{T}\end{array}\right) u$,
where $y=\left(\begin{array}{c}y_{1} \\ y_{2} \\ \vdots \\ y_{n}\end{array}\right), X\left(\begin{array}{c}X_{1} \\ X_{2} \\ \vdots \\ X_{n}\end{array}\right)$, and $u=\left(\begin{array}{c}u_{1} \\ u_{2} \\ \vdots \\ u_{n}\end{array}\right)$, which are $\operatorname{Tn} \times 1, T n \times k$ and $T n \times 1$ matrices, respectively

Using the Kronecker product, we obtain the following expression:

$$
\left(I_{n} \otimes D_{T}\right) y=\left(I_{n} \otimes D_{T}\right) X \beta+\left(I_{n} \otimes D_{T}\right) u
$$

where $\left(I_{n} \otimes D_{T}\right), y, X$, and $u$ are $n T \times n T, n T \times 1, n T \times k$, and $n T \times 1$, respectively.

## Kronecker Product - Review:

1. $A: n \times m, \quad B: T \times k$

$$
A \otimes B=\left(\begin{array}{cccc}
a_{11} B & a_{12} B & \cdots & a_{1 m} B \\
a_{21} B & a_{22} B & \cdots & a_{2 m} B \\
\vdots & \vdots & \cdots & \vdots \\
a_{n 1} B & a_{n 2} B & \cdots & a_{n m} B
\end{array}\right) \text {, which is a } n T \times m k \text { matrix. }
$$

2. $A: n \times n, \quad B: m \times m$

$$
\begin{array}{lr}
(A \otimes B)^{-1}=A^{-1} \otimes B^{-1}, & |A \otimes B|=|A|^{m}|B|^{n}, \\
(A \otimes B)^{\prime}=A^{\prime} \otimes B^{\prime}, & \operatorname{tr}(A \otimes B)=\operatorname{tr}(A) \operatorname{tr}(B) .
\end{array}
$$

3. For $A, B, C$ and $D$ such that the products are defined,

$$
(A \otimes B)(C \otimes D)=A C \otimes B D .
$$

End of Review

Going back to the previous slide, using the Kronecker product, we obtain the following expression:

$$
\left(I_{n} \otimes D_{T}\right) y=\left(I_{n} \otimes D_{T}\right) X \beta+\left(I_{n} \otimes D_{T}\right) u,
$$

where $\left(I_{n} \otimes D_{T}\right), y, X$, and $u$ are $n T \times n T, n T \times 1, n T \times k$, and $n T \times 1$, respectively.

Apply OLS to the above regression model.

$$
\begin{aligned}
\hat{\beta} & =\left(\left(\left(I_{n} \otimes D_{T}\right) X\right)^{\prime}\left(I_{n} \otimes D_{T}\right) X\right)^{-1}\left(\left(I_{n} \otimes D_{T}\right) X\right)^{\prime}\left(I_{n} \otimes D_{T}\right) y \\
& =\left(X^{\prime}\left(I_{n} \otimes D_{T}^{\prime} D_{T}\right) X\right)^{-1} X^{\prime}\left(I_{n} \otimes D_{T}^{\prime} D_{T}\right) y \\
& =\left(X^{\prime}\left(I_{n} \otimes D_{T}\right) X\right)^{-1} X^{\prime}\left(I_{n} \otimes D_{T}\right) y .
\end{aligned}
$$

Note that the inverse matrix of $D_{T}$ is not available, because the rank of $D_{T}$ is $T-1$, not $T$ (full rank).

The rank of a symmetric and idempotent matrix is equal to its trace.

The fixed effect $v_{i}$ is estimated as:

$$
\hat{v}_{i}=\bar{y}_{i}-\bar{X}_{i} \hat{\beta} .
$$

Possibly, we can estimate the following regression:

$$
\hat{v}_{i}=Z_{i} \alpha+\epsilon_{i},
$$

where it is assumed that the individual-specific effect depends on $Z_{i}$.

The estimator of $\sigma_{u}^{2}$ is given by:

$$
\hat{\sigma}_{u}^{2}=\frac{1}{n T-k-n} \sum_{i=1}^{n} \sum_{t=1}^{T}\left(y_{i t}-X_{i t} \hat{\beta}-\hat{v}_{i}\right)^{2} .
$$

## [Remark]

More than ten years ago, "fixed" indicates that $v_{i}$ is nonstochastic.
Recently, however, "fixed" does not mean anything.
"fixed" indicates that OLS is applied and that $v_{i}$ may be correlated with $X_{i t}$.

Possibly, $\mathrm{E}\left(v_{i} \mid X\right)=\alpha_{i}(X)$, where $\alpha_{i}(X)$ is a function of $X_{i t}$ for $i=1,2, \cdots, n$ and $t=1,2, \cdots, T$, and it is normalized to $\sum_{i=1}^{n} \alpha_{i}(X)=0$.

