i.e.,

$$\begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} y = \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} X \beta + \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} u,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, X \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, which are $Tn \times 1$, $Tn \times k$ and $Tn \times 1$ matrices, respectively

Using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Kronecker Product — **Review:**

1. A:
$$n \times m$$
, B: $T \times k$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}$$
, which is a $nT \times mk$ matrix.
2. A: $n \times n$, B: $m \times m$

 $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad |A \otimes B| = |A|^m |B|^n,$ $(A \otimes B)' = A' \otimes B', \qquad \operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B).$

3. For A, B, C and D such that the products are defined,

 $(A \otimes B)(C \otimes D) = AC \otimes BD.$

End of Review

Going back to the previous slide, using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Apply OLS to the above regression model.

$$\hat{\beta} = \left(((I_n \otimes D_T)X)'(I_n \otimes D_T)X \right)^{-1} ((I_n \otimes D_T)X)'(I_n \otimes D_T)y$$
$$= \left(X'(I_n \otimes D'_T D_T)X \right)^{-1} X'(I_n \otimes D'_T D_T)y$$
$$= \left(X'(I_n \otimes D_T)X \right)^{-1} X'(I_n \otimes D_T)y.$$

Note that the inverse matrix of D_T is not available, because the rank of D_T is T - 1, not T (full rank).

The rank of a symmetric and idempotent matrix is equal to its trace.

The fixed effect v_i is estimated as:

$$\hat{v}_i = \overline{y}_i - \overline{X}_i \hat{\beta}.$$

Possibly, we can estimate the following regression:

$$\hat{v}_i = Z_i \alpha + \epsilon_i,$$

where it is assumed that the individual-specific effect depends on Z_i .

The estimator of σ_u^2 is given by:

$$\hat{\sigma}_{u}^{2} = \frac{1}{nT - k - n} \sum_{i=1}^{n} \sum_{t=1}^{T} (y_{it} - X_{it}\hat{\beta} - \hat{v}_{i})^{2}.$$

[Remark]

More than ten years ago, "fixed" indicates that v_i is nonstochastic.

Recently, however, "fixed" does not mean anything.

"fixed" indicates that OLS is applied and that v_i may be correlated with X_{it} .

Possibly, $E(v_i|X) = \alpha_i(X)$, where $\alpha_i(X)$ is a function of X_{it} for $i = 1, 2, \dots, n$ and $t = 1, 2, \dots, T$, and it is normalized to $\sum_{i=1}^{n} \alpha_i(X) = 0$.