14 Panel Data

14.1 GLS — Review

Regression model:

$$y = X\beta + u, \qquad u \sim N(0, \Omega),$$

where y, X, β, u, 0 and Ω are $n \times 1$, $n \times k$, $k \times 1$, $n \times 1$, $n \times 1$, and $n \times n$, respectively.

We solve the following minimization problem:

$$\min_{\beta} (y - X\beta)'\Omega^{-1}(y - X\beta).$$

Let b be a solution of the above minimization problem.

GLS estimator of β is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

14.2 Panel Model Basic

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where i indicates individual and t denotes time.

There are n observations for each t.

u_{it} indicates the error term, assuming that $\mathrm{E}(u_{it}) = 0$, $\mathrm{V}(u_{it}) = \sigma_u^2$ and $\mathrm{Cov}(u_{it}, u_{js}) = 0$ for $i \neq j$ and $t \neq s$.

v_i denotes the individual effect, which is fixed or random.

14.2.1 Fixed Effect Model (固定効果モデル)

In the case where v_i is fixed, the case of $v_i = z_i\alpha$ is included.

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$
$$\overline{y}_i = \overline{X}_i\beta + v_i + \overline{u}_i, \qquad i = 1, 2, \cdots, n,$$

where $\overline{y}_i = \dfrac{1}{T}\sum_{t=1}^{T} y_{it}$, $\overline{X}_i = \dfrac{1}{T}\sum_{t=1}^{T} X_{it}$, and $\overline{u}_i = \dfrac{1}{T}\sum_{t=1}^{T} u_{it}$.

$$(y_{it} - \overline{y}_i) = (X_{it} - \overline{X}_i)\beta + (u_{it} - \overline{u}_i), \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

Taking an example of y, the left-hand side of the above equation is rewritten as:

$$y_{it} - \overline{y}_i = y_{it} - \frac{1}{T}1_T' y_i,$$

where $1_T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, which is a $T \times 1$ vector, and $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$.

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = I_T y_i - 1_T \overline{y}_i = I_T y_i - \frac{1}{T} 1_T 1_T' y_i = (I_T - \frac{1}{T} 1_T 1_T') y_i$$

Thus,

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = \begin{pmatrix} X_{i1} - \overline{X}_i \\ X_{i2} - \overline{X}_i \\ \vdots \\ X_{iT} - \overline{X}_i \end{pmatrix} \beta + \begin{pmatrix} u_{i1} - \overline{u}_i \\ u_{i2} - \overline{u}_i \\ \vdots \\ u_{iT} - \overline{u}_i \end{pmatrix}, \qquad i = 1, 2, \cdots, n,$$

which is re-written as:

$$(I_T - \frac{1}{T} 1_T 1_T') y_i = (I_T - \frac{1}{T} 1_T 1_T') X_i \beta + (I_T - \frac{1}{T} 1_T 1_T') u_i, \qquad i = 1, 2, \cdots, n,$$

i.e.,

$$D_T y_i = D_T X_i \beta + D_T u_i, \qquad i = 1, 2, \cdots, n,$$

where $D_T = (I_T - \frac{1}{T} 1_T 1_T')$, which is a $T \times T$ matrix.

Note that $D_T D_T' = D_T$, i.e., D_T is a symmetric and idempotent matrix.

Using the matrix form for $i = 1, 2, \cdots, n$, we have:

$$\begin{pmatrix} D_T y_1 \\ D_T y_2 \\ \vdots \\ D_T y_n \end{pmatrix} = \begin{pmatrix} D_T X_1 \\ D_T X_2 \\ \vdots \\ D_T X_n \end{pmatrix} \beta + \begin{pmatrix} D_T u_1 \\ D_T u_2 \\ \vdots \\ D_T u_n \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} y = \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} X\beta + \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} u,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $X \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, which are $Tn \times 1$, $Tn \times k$ and $Tn \times 1$ matrices, respectively

Using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Kronecker Product — Review:

1. A: $n \times m$, B: $T \times k$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}, \text{ which is a } nT \times mk \text{ matrix.}$$

2. A: $n \times n$, B: $m \times m$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad |A \otimes B| = |A|^m |B|^n,$$

$$(A \otimes B)' = A' \otimes B', \qquad \mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B).$$

3. For A, B, C and D such that the products are defined,

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

End of Review

Going back to the previous slide, using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Apply OLS to the above regression model.

$$\begin{aligned}
\hat{\beta} &= \Big(((I_n \otimes D_T)X)'(I_n \otimes D_T)X\Big)^{-1}((I_n \otimes D_T)X)'(I_n \otimes D_T)y \\
&= \Big(X'(I_n \otimes D_T'D_T)X\Big)^{-1}X'(I_n \otimes D_T'D_T)y \\
&= \Big(X'(I_n \otimes D_T)X\Big)^{-1}X'(I_n \otimes D_T)y.
\end{aligned}$$

Note that the inverse matrix of D_T is not available, because the rank of D_T is $T-1$, not T (full rank).

The rank of a symmetric and idempotent matrix is equal to its trace.

The fixed effect v_i is estimated as:

$$\hat{v}_i = \overline{y}_i - \overline{X}_i\hat{\beta}.$$

Possibly, we can estimate the following regression:

$$\hat{v}_i = Z_i\alpha + \epsilon_i,$$

where it is assumed that the individual-specific effect depends on Z_i.

The estimator of σ_u^2 is given by:

$$\hat{\sigma}_u^2 = \frac{1}{nT - k - n}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it} - X_{it}\hat{\beta} - \hat{v}_i)^2.$$

[Remark]

More than ten years ago, "fixed" indicates that v_i is nonstochastic.
Recently, however, "fixed" does not mean anything.
"fixed" indicates that OLS is applied and that v_i may be correlated with X_{it}.

Possibly, $\mathrm{E}(v_i|X) = \alpha_i(X)$, where $\alpha_i(X)$ is a function of X_{it} for $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$, and it is normalized to $\sum_{i=1}^{n} \alpha_i(X) = 0$.

14.2.2 Random Effect Model (ランダム効果モデル)

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where i indicates individual and t denotes time.

The assumptions on the error terms v_i and u_{it} are:

$$\mathrm{E}(v_i|X) = \mathrm{E}(u_{it}|X) = 0 \ \text{ for all } i,$$
$$\mathrm{V}(v_i|X) = \sigma_v^2 \ \text{ for all } i, \qquad \mathrm{V}(u_{it}|X) = \sigma_u^2 \ \text{ for all } i \text{ and } t,$$
$$\mathrm{Cov}(v_i, v_j|X) = 0 \ \text{ for } i \neq j, \qquad \mathrm{Cov}(u_{it}, u_{js}|X) = 0 \ \text{ for } i \neq j \text{ and } t \neq s,$$
$$\mathrm{Cov}(v_i, u_{jt}|X) = 0 \ \text{ for all } i,\ j \text{ and } t.$$

Note that X includes X_{it} for $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

In a matrix form with respect to $t = 1, 2, \cdots, T$, we have the following:

$$y_i = X_i\beta + v_i 1_T + u_i, \qquad i = 1, 2, \cdots, n,$$

where $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$, $X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{iT} \end{pmatrix}$ and $u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{pmatrix}$ are $T \times 1$, $T \times k$ and $T \times 1$, respectively.

$$u_i \sim N(0, \sigma_u^2 I_T) \text{ and } v_i 1_T \sim N(0, \sigma_v^2) \implies v_i 1_T + u_i \sim N(0, \sigma_v^2 1_T 1_T' + \sigma_u^2 I_T).$$

Again, in a matrix form with respect to i, we have the following:

$$y = X\beta + v + u,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, $v = \begin{pmatrix} v_1 1_T \\ v_2 1_T \\ \vdots \\ v_n 1_T \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ are $nT \times 1$, $nT \times k$, $nT \times 1$ and $nT \times 1$, respectively.

The distribution of $u + v$ is given by:

$$v + u \sim N\Big(0, I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)$$

The likelihood function is given by:

$$L(\beta, \sigma_v^2, \sigma_u^2) = (2\pi)^{-nT/2} \left| I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T) \right|^{-1/2} \times \exp\Big(-\frac{1}{2} (y - X\beta)' \big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T) \big)^{-1} (y - X\beta) \Big).$$

Remember that $f(x) = (2\pi)^{-k/2} |\Sigma|^{-1/2} \exp\Big(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big)$ when $X \sim N(\mu, \Sigma)$, where X denotes a k-variate random variable.

The estimators of β, σ_v^2 and σ_u^2 are given by maximizing the following log-likelihood function:

$$\log L(\beta, \sigma_v^2, \sigma_u^2) = -\frac{nT}{2}\log(2\pi) - \frac{1}{2}\log\Big|I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big| \\ -\frac{1}{2}(y - X\beta)'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1}(y - X\beta).$$

MLE of β, denoted by $\tilde{\beta}$, is given by:

$$\tilde{\beta} = \Big(X'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1} X\Big)^{-1} X'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1} y \\ = \Big(\sum_{i=1}^{n} X_i'(\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)^{-1} X_i\Big)^{-1}\Big(\sum_{i=1}^{n} X_i'(\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)^{-1} y_i\Big),$$

which is equivalent to GLS.

Note that $\tilde{\beta}$ is not operational, because $\hat{\beta}$ depends on σ_v^2 and σ_u^2.

14.3 Hausman's Specification Error (特定化誤差) Test

Regression model:

$$y = X\beta + u, \qquad y:\ n\times 1, \quad X:\ n\times k, \quad \beta:\ k\times 1, \quad u:\ n\times 1.$$

Suppose that X is stochastic.

If $\mathrm{E}(u|X) = 0$, OLSE $\hat{\beta}$ is unbiased because of $\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$ and $\mathrm{E}((X'X)^{-1}X'u) = 0$.

However, If $\mathrm{E}(u|X) \neq 0$, OLSE $\hat{\beta}$ is biased and inconsistent.

Therefore, we need to check if X is correlated with u or not.
$\Longrightarrow$ **Hausman's Specification Error Test**

The null and alternative hypotheses are:

- H_0 : X and u are independent, i.e., $\mathrm{Cov}(X, u) = 0$,
- H_1 : X and u are not independent.

Suppose that we have two estimators $\hat{\beta}_0$ and $\hat{\beta}_1$, which have the following properties:

- $\hat{\beta}_0$ is consistent and efficient under H_0, but is not consistent under H_1,
- $\hat{\beta}_1$ is consistent under both H_0 and H_1, but is not efficient under H_0.

Under the conditions above, we have the following test statistic:

$$(\hat{\beta}_1 - \hat{\beta}_0)'\Big(\mathrm{V}(\hat{\beta}_1) - \mathrm{V}(\hat{\beta}_0)\Big)^{-1}(\hat{\beta}_1 - \hat{\beta}_0) \longrightarrow \chi^2(k).$$

Example: $\hat{\beta}_0$ is OLS, while $\hat{\beta}_1$ is IV such as 2SLS.

Hausman, J.A. (1978) “Specification Tests in Econometrics,” *Econometrica*, Vol.46, No.6, pp.1251–1271.

14.4 Choice of Fixed Effect Model or Random Effect Model

14.4.1 The Case where X is Correlated with u — Review

The standard regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I_n)$$

OLS is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u.$$

If X is not correlated with u, i.e., $\mathrm{E}(X'u) = 0$, we have the result: $\mathrm{E}(\hat{\beta}) = \beta$.

However, if X is correlated with u, i.e., $\mathrm{E}(X'u) \neq 0$, we have the result: $\mathrm{E}(\hat{\beta}) \neq \beta$.
$\Longrightarrow$ $\hat{\beta}$ is biased.

Assume that in the limit we have the followings:

$$(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1},$$
$$\frac{1}{n}X'u \longrightarrow M_{xu} \neq 0 \text{ when } X \text{ is correlated with } u.$$

Therefore, even in the limit,

$$\text{plim}\,\hat{\beta} = \beta + M_{xx}^{-1}M_{xu} \neq \beta,$$

which implies that $\hat{\beta}$ is not a consistent estimator of β.

Thus, in the case where X is correlated with u, OLSE $\hat{\beta}$ is neither unbiased nor consistent.

14.4.2 Fixed Effect Model or Random Effect Model

Usually, in the random effect model, we can consider that v_i is correlated with X_{it}.

[Reason:]

v_i includes the unobserved variables in the ith individual, i.e., ability, intelligence, and so on.

X_{it} represents the observed variables in the ith individual, i.e., income, assets, and so on.

The unobserved variables v_i are related to the observed variables X_{it}.

Therefore, we consider that v_i is correlated with X_{it}.

Thus, in the case of the random effect model, usually we cannot use OLS or GLS.

In order to use the random effect model, we need to test whether v_i is uncorrelated with X_{it}.

Apply Hausman's test.

- H_0 : X_{it} and e_{it} are independent ($\longrightarrow$ Use the random effect model),
- H_1 : X_{it} and e_{it} are not independent ($\longrightarrow$ Use the fixed effect model),

where $e_{it} = v_i + u_{it}$.

Example of Panel Data:
Production Function of Prefectures from 2001 to 2010.

pref: 都道府県（通し番号 1～47）
year: 年度（2001～2010 年）
y　: 県内総生産（支出側、実質 : 固定基準年方式），出所 : 県民経済計算（平成 13 年度 - 平成 24 年度）（93SNA，平成 17 年基準計数）
k　: 都道府県別民間資本ストック（平成 12 暦年価格，年度末，国民経済計算ベース　平成 23 年 3 月時点）一期前（2000～2009 年）
l　: 県内就業者数，出所 : 県民経済計算（平成 13 年度 - 平成 24 年度）（93SNA，平成 17 年基準計数）

```
. tsset pref year
       panel variable:  pref (strongly balanced)
        time variable:  year, 2001 to 2010
                delta:  1 unit

. gen ly=log(y)

. gen lk=log(k)

. gen ll=log(l)
```

```
. reg ly lk ll

      Source |       SS       df       MS              Number of obs =     470
-------------+------------------------------           F(  2,   467) =19374.95
       Model |  316.479302     2  158.239651           Prob > F      =  0.0000
    Residual |  3.81409572   467  .008167229           R-squared     =  0.9881
-------------+------------------------------           Adj R-squared =  0.9880
       Total |  320.293398   469  .682928354           Root MSE      =  .09037

------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          lk |   .0941587   .0081273    11.59   0.000     .0781881    .1101294
          ll |   .9976399   .0102641    97.20   0.000     .9774703    1.017809
       _cons |   .5970719   .0773137     7.72   0.000     .4451461    .7489978
------------------------------------------------------------------------------

. xtreg ly lk ll,fe

Fixed-effects (within) regression               Number of obs      =       470
Group variable: pref                            Number of groups   =        47

R-sq:  within  = 0.1721                         Obs per group: min =        10
       between = 0.9456                                        avg =      10.0
       overall = 0.9439                                        max =        10

                                                F(2,421)           =     43.77
corr(u_i, Xb)  = 0.8803                         Prob > F           =    0.0000
```

```
------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          lk |   .2329208   .0252321     9.23   0.000     .1833242    .2825175
          ll |   .3268537   .0810662     4.03   0.000     .1675088    .4861987
       _cons |   7.691145   1.376677     5.59   0.000     4.985128    10.39716
-------------+----------------------------------------------------------------
     sigma_u |  .41045507
     sigma_e |  .03561437
         rho |  .99252757   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(46, 421) =    56.22             Prob > F = 0.0000

. est store fixed

. xtreg ly lk ll,re

Random-effects GLS regression                   Number of obs      =       470
Group variable: pref                            Number of groups   =        47

R-sq:  within  = 0.1058                         Obs per group: min =        10
       between = 0.9805                                        avg =      10.0
       overall = 0.9787                                        max =        10

                                                Wald chi2(2)       =   3875.75
corr(u_i, X)   = 0 (assumed)                    Prob > chi2        =    0.0000

------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
```

```
-------------+----------------------------------------------------------------
          lk |   .2457767   .0153094    16.05   0.000     .2157708    .2757827
          ll |   .8105099   .0220256    36.80   0.000     .7673406    .8536793
       _cons |   .8332015   .2411141     3.46   0.001     .3606265    1.305776
-------------+----------------------------------------------------------------
     sigma_u |    .081609
     sigma_e |  .03561437
         rho |   .8400205   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. hausman fixed

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |     fixed          .          Difference          S.E.
-------------+----------------------------------------------------------------
          lk |    .2329208     .2457767       -.0128559         .020057
          ll |    .3268537     .8105099       -.4836562        .0780167
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from xtreg
            B = inconsistent under Ha, efficient under Ho; obtained from xtreg

    Test:  Ho:  difference in coefficients not systematic

                  chi2(2) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =      144.66
                Prob>chi2 =      0.0000
```