[Review] Formulas — F Distribution:

- $\dfrac{U/n}{V/m} \sim F(n,m)$ when U $sim \chi^2(n)$, $V \sim \chi^2(m)$, and U is independent of V.

- When $X \sim N(0, I_n)$, A and B are $n \times n$ symmetric idempotent matrices, $\text{Rank}(A) = \text{tr}(A) = G$, $\text{Rank}(B) = \text{tr}(B) = K$ and $AB = 0$, then $\dfrac{X'AX/G}{X'BX/K} \sim F(G,K)$.

Note that the covariance of AX and BX is zero, which implies that AX is independent of BX under normality of X.

[End of Review]

6. Therefore, we obtain the following distribution:

$$\frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k),$$

$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(n-k)$$

$\hat\beta$ is independent of e, because $X(X'X)^{-1}X'(I_n - X(X'X)^{-1}X') = 0$.

Accordingly, we can derive:

$$\frac{\dfrac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)/k}{s^2} \sim F(k, n-k)$$

Under the null hypothesis $H_0:\ \beta = 0$, $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2} \sim F(k, n-k)$.

Given data, $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2}$ is compared with $F(k, n-k)$.

If $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2}$ is in tha tail of the F distribution, the null hypothesis is rejected.

Coefficient of Determination (決定係数), R^2:

1. Definition of the Coefficient of Determination, R^2: $\quad R^2 = 1 - \dfrac{\sum_{i=1}^n e_i^2}{\sum_{i=1}^n (y_i - \overline{y})^2}$

2. Numerator: $\quad \displaystyle\sum_{i=1}^n e_i^2 = e'e$

3. Denominator: $\quad \displaystyle\sum_{i=1}^n (y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')'(I_n - \frac{1}{n}ii')y = y'(I_n - \frac{1}{n}ii')y$

(*) Remark

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{pmatrix} = y - \frac{1}{n}ii'y = (I_n - \frac{1}{n}ii')y,$$

where $i = (1, 1, \cdots, 1)'$.

4. In a matrix form, we can rewrite as: $R^2 = 1 - \dfrac{e'e}{y'(I_n - \frac{1}{n}ii')y}$

F Distribution and Coefficient of Determination:

$\Longrightarrow$ This will be discussed later.

Testing Linear Restrictions (F Distribution):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R: \; G \times k, \qquad \text{rank}(R) = G \leq k.$

   $R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$.

   Therefore, $$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(G).$$

   Note that $R\beta = r$.

   (a) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the mean of $R\hat{\beta}$ is:

   $$\mathrm{E}(R\hat{\beta}) = R\mathrm{E}(\hat{\beta}) = R\beta.$$

   (b) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the variance of $R\hat{\beta}$ is:

$$V(R\hat{\beta}) = E((R\hat{\beta} - R\beta)(R\hat{\beta} - R\beta)') = E(R(\hat{\beta} - \beta)(\hat{\beta} - \beta)'R')$$
$$= RE((\hat{\beta} - \beta)(\hat{\beta} - \beta)')R' = RV(\hat{\beta})R' = \sigma^2 R(X'X)^{-1}R'.$$

2. We know that $\dfrac{(n-k)s^2}{\sigma^2} = \dfrac{e'e}{\sigma^2} = \dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{\sigma^2} \sim \chi^2(n-k)$.

3. Under normality assumption on u, $\hat{\beta}$ is independent of e.

4. Therefore, we have the following distribution:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(G, n-k)$$

5. **Some Examples:**

(a) t Test:

The case of $G = 1$, $r = 0$ and $R = (0, \cdots, 1, \cdots, 0)$ (the ith element of R is one and the other elements are zero):

The test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{s^2} = \frac{\hat{\beta}_i^2}{s^2 a_{ii}} \sim F(1, n-k),$$

where $s^2 = e'e/(n-k)$, $R\hat{\beta} = \hat{\beta}_i$ and

$$a_{ii} = R(X'X)^{-1}R' = \text{ the } i \text{ row and } i\text{th column of } (X'X)^{-1}.$$

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{\hat{\beta}_i}{s\sqrt{a_{ii}}} \sim t(n-k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m+1, m+2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the F test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \text{rank}(R) = k$ and β is a $2k \times 1$ vector.

The distribution is $F(k, n - 2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \text{rank}(R) = 1$

The distribution of the test statistic is $F(1, n-k)$.

(d) Testing seasonality:

In the case of **quarterly data** (四半期データ), the regression model is:

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j = 1$ in the jth quarter and 0 otherwise, i.e., D_j, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\Longrightarrow H_0: \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \text{rank}(R) = 3$, and β is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n-k)$.

(e) Cobb-Douglas Production Function:

Let Q_i, K_i and L_i be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We test a linear homogeneous (一次同次) production function.

The null and alternative hypotheses are:

$$H_0 : \ \beta_2 + \beta_3 = 1,$$

$$H_1 : \ \beta_2 + \beta_3 \neq 1.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods m and $m+1$.

In the case where both the constant term and the slope are changed, the regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m+1, m+2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m+1$.

The null and alternative hypotheses are as follows:

$H_0:\ \gamma = \delta = 0,$

$H_1:\ \gamma \neq 0$, or, $\delta \neq 0$.

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither x_i nor z_i depends on y_i.

In this case, the null and alternative hypotheses are as follows:

$$H_0 : \beta = \gamma = 0,$$

$$H_1 : \beta \neq 0, \text{ or, } \gamma \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$