Coefficient of Determination R^2 and F distribution:

- The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = \begin{pmatrix} 1 & x_{2i} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$$x_i : \ 1 \times k, \qquad x_{2i} : \ 1 \times (k-1), \qquad \beta : \ k \times 1, \qquad \beta_2 : \ (k-1) \times 1$$

Define:

$$X_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Then,

$$y = X\beta + u = \begin{pmatrix} i & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u = i\beta_1 + X_2\beta_2 + u,$$

where the first column of X corresponds to a constant term, i.e.,

$$X = (\, i \quad X_2 \,), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- Consider testing $H_0: \ \beta_2 = 0$.

The F distribution is set as follows:

$$R = (\, 0 \quad I_{k-1} \,), \qquad r = 0$$

where R is a $(k-1) \times k$ matrix and r is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

We are going to show:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \frac{1}{n} i i'$.

Note that M is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$\begin{aligned} R(X'X)^{-1}R' &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} \begin{pmatrix} i & X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \end{aligned}$$

[Review] The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.

[End of Review]

Go back to the F distribution.

$$\begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix}$$

Therefore, we obtain:

$$\begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}.$$

Thus, under $H_0:\ \beta_2 = 0$, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k).$$

● Coefficient of Determination R^2:

Define e as $e = y - X\hat{\beta}$. The coefficient of determinant, R^2, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \frac{1}{n}ii'$, I_n is a $n \times n$ identity matrix and i is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = (\, i \quad X_2 \,)$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

because $i'e = 0$, and

$$MX = M(\, i \quad X_2 \,) = (\, Mi \quad MX_2 \,) = (\, 0 \quad MX_2 \,),$$

because $Mi = 0$.

$$MX\hat{\beta} = \begin{pmatrix} 0 & MX_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = MX_2\hat{\beta}_2.$$

Thus,

$$My = MX\hat{\beta} + Me \qquad \Longrightarrow \qquad My = MX_2\hat{\beta}_2 + e.$$

$y'My$ is given by: $y'My = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 + e'e$, because $X_2'e = 0$ and $Me = e$.

The coefficient of determinant, R^2, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \qquad \Longrightarrow \qquad e'e = (1 - R^2)y'My,$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My} \qquad \Longrightarrow \qquad \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 = R^2y'My.$$

Therefore,

$$\frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} = \frac{R^2y'My/(k-1)}{(1-R^2)y'My/(n-k)} = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k).$$

Thus, using R^2, the null hypothesis $H_0 : \beta_2 = 0$ is easily tested.

5 Restricted OLS (制約付き最小二乗法)

1. Let $\tilde{\beta}$ be the restricted estimator.

   Consider the linear restriction: $R\beta = r$.

2. Minimize $(y - X\tilde{\beta})'(y - X\tilde{\beta})$ subject to $R\tilde{\beta} = r$.

   Let L be the Lagrangian for the minimization problem.

$$L = (y - X\tilde{\beta})'(y - X\tilde{\beta}) - 2\tilde{\lambda}'(R\tilde{\beta} - r)$$

   Because $\tilde{\beta}$ and $\tilde{\lambda}$ minimize the Lagrangian L,

$$\frac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0$$
$$\frac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0.$$

(*) Remember that $\dfrac{\partial a'x}{\partial x} = a$ and $\dfrac{\partial x'Ax}{\partial x} = (A + A')x$.

From $\dfrac{\partial L}{\partial \tilde{\beta}} = 0$, we obtain:

$$\tilde{\beta} = (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}.$$

Multiplying R from the left, we have:

$$R\tilde{\beta} = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Because $R\tilde{\beta} = r$ has to be satisfied, we have the following expression:

$$r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Therefore, solving the above equation with respect to $\tilde{\lambda}$, we obtain:

$$\tilde{\lambda} = \left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta})$$

Substituting $\tilde{\lambda}$ into $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}$, the restricted OLSE is given by:

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}).$$

(a) The expectation of $\tilde{\beta}$ is:

$$\begin{aligned} \mathrm{E}(\tilde{\beta}) &= \mathrm{E}(\hat{\beta}) + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\mathrm{E}(\hat{\beta})) \\ &= \beta + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\beta) \\ &= \beta, \end{aligned}$$

because of $R\beta = r$.

Thus, it is shown that $\tilde{\beta}$ is unbiased.

(b) The variance of $\tilde{\beta}$ is as follows.

First, rewrite as follows:

$$
\begin{aligned}
(\tilde{\beta} - \beta) &= (\hat{\beta} - \beta) + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\beta - R\hat{\beta}) \\
&= (\hat{\beta} - \beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - R\beta) \\
&= (\hat{\beta} - \beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(\hat{\beta} - \beta) \\
&= \left(I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\right)(\hat{\beta} - \beta) \\
&= W(\hat{\beta} - \beta),
\end{aligned}
$$

where $W \equiv I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R$.

Then, we obtain the following variance:

$$
\begin{aligned}
\mathrm{V}(\tilde{\beta}) &\equiv \mathrm{E}((\tilde{\beta} - \beta)(\tilde{\beta} - \beta)') = \mathrm{E}(W(\hat{\beta} - \beta)(\hat{\beta} - \beta)'W') \\
&= W\mathrm{E}((\hat{\beta} - \beta)(\hat{\beta} - \beta)')W' = W\mathrm{V}(\hat{\beta})W' = \sigma^2 W(X'X)^{-1}W'
\end{aligned}
$$

$$= \sigma^2\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)(X'X)^{-1}$$

$$\times\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)'$$

$$= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}$$

$$= \mathrm{V}(\hat{\beta}) - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}$$

Thus, $\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta})$ is positive definite.