

2. **Central Limit Theorem:** Greenberg and Webster (1983)

Z_1, Z_2, \dots, Z_n are mutually independently distributed with mean μ and variance Σ_i .

Then, we have the following result:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_i - \mu) \longrightarrow N(0, \Sigma),$$

where

$$\Sigma = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \Sigma_i \right).$$

The distribution of Z_i is not assumed.

3. Define $Z_i = x'_i u_i$. Then, $\Sigma_i = \text{Var}(Z_i) = \sigma^2 x'_i x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \sigma^2 x_i' x_i \right) = \sigma^2 \lim_{n \rightarrow \infty} \left(\frac{1}{n} X' X \right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983)), we obtain the following:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i' u_i = \frac{1}{\sqrt{n}} X' u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X' X)^{-1} X' u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n} X' X \right)^{-1} \frac{1}{\sqrt{n}} X' u.$$

$$\begin{aligned}
\text{Var}\left(\left(\frac{1}{n}X'X\right)^{-1} \frac{1}{\sqrt{n}}X'u\right) &= E\left(\left(\frac{1}{n}X'X\right)^{-1} \frac{1}{\sqrt{n}}X'u\left(\left(\frac{1}{n}X'X\right)^{-1} \frac{1}{\sqrt{n}}X'u\right)'\right) \\
&= \left(\frac{1}{n}X'X\right)^{-1} \left(\frac{1}{n}X'E(uu')X\right) \left(\frac{1}{n}X'X\right)^{-1} \\
&= \sigma^2 \left(\frac{1}{n}X'X\right)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}.
\end{aligned}$$

Therefore,

$$\sqrt{n}(\hat{\beta} - \beta) \longrightarrow N(0, \sigma^2 M_{xx}^{-1})$$

\implies Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

12 Instrumental Variable (操作変数法)

12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

V : is called the **measurement error** (測定誤差 or 観測誤差).

3. For the elements which do not include measurement errors in X , the corresponding elements in V are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of β is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

(a) The measurement error in X is uncorrelated with \tilde{X} in the limit. i.e.,

$$\text{plim}\left(\frac{1}{n}\tilde{X}'V\right) = 0.$$

Therefore, we obtain the following:

$$\text{plim}\left(\frac{1}{n}X'X\right) = \text{plim}\left(\frac{1}{n}\tilde{X}'\tilde{X}\right) + \text{plim}\left(\frac{1}{n}V'V\right) = \Sigma + \Omega$$

(b) u is not correlated with V .

u is not correlated with \tilde{X} .

That is,

$$\text{plim}\left(\frac{1}{n}V'u\right) = 0, \quad \text{plim}\left(\frac{1}{n}\tilde{X}'u\right) = 0.$$

6. OLSE of β is:

$$\hat{\beta} = \beta + (X'X)^{-1}X'(u - V\beta) = \beta + (X'X)^{-1}(\tilde{X} + V)'(u - V\beta).$$

Therefore, we obtain the following:

$$\text{plim} \hat{\beta} = \beta - (\Sigma + \Omega)^{-1}\Omega\beta$$

7. Example: The Case of Two Variables:

The regression model is given by:

$$y_t = \alpha + \beta \tilde{x}_t + u_t, \quad x_t = \tilde{x}_t + v_t.$$

Under the above model,

$$\Sigma = \text{plim}\left(\frac{1}{n}\tilde{X}'\tilde{X}\right) = \text{plim}\left(\begin{array}{cc} 1 & \frac{1}{n}\sum\tilde{x}_i \\ \frac{1}{n}\sum\tilde{x}_i & \frac{1}{n}\sum\tilde{x}_i^2 \end{array}\right) = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix},$$

where μ and σ^2 represent the mean and variance of \tilde{x}_i .

$$\Omega = \text{plim}\left(\frac{1}{n}V'V\right) = \text{plim}\left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{n}\sum v_i^2 \end{array}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \text{plim}\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \left(\begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \frac{1}{\sigma^2 + \sigma_v^2} \begin{pmatrix} -\mu\sigma_v^2\beta \\ \sigma_v^2\beta \end{pmatrix} \end{aligned}$$

Now we focus on β .

$\hat{\beta}$ is not consistent. because of:

$$\text{plim}(\hat{\beta}) = \beta - \frac{\sigma_v^2 \beta}{\sigma^2 + \sigma_v^2} = \frac{\beta}{1 + \sigma_v^2/\sigma^2} < \beta$$

12.2 Instrumental Variable (IV) Method (操作変数法 or IV 法)

Instrumental Variable (IV)

1. Consider the regression model: $y = X\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

In the case of $E(X'u) \neq 0$, OLSE of β is inconsistent.

2. Proof:

$$\hat{\beta} = \beta + \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{n}X'u \longrightarrow \beta + M_{xx}^{-1}M_{xu},$$

where

$$\frac{1}{n}X'X \longrightarrow M_{xx}, \quad \frac{1}{n}X'u \longrightarrow M_{xu} \neq 0$$

3. Find the Z which satisfies $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

Multiplying Z' on both sides of the regression model: $y = X\beta + u$,

$$Z'y = Z'X\beta + Z'u$$

Dividing n on both sides of the above equation, we take plim on both sides.

Then, we obtain the following:

$$\text{plim}\left(\frac{1}{n}Z'y\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta + \text{plim}\left(\frac{1}{n}Z'u\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta.$$

Accordingly, we obtain:

$$\beta = \left(\text{plim}\left(\frac{1}{n}Z'X\right)\right)^{-1} \text{plim}\left(\frac{1}{n}Z'y\right).$$

Therefore, we consider the following estimator:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

which is taken as an estimator of β .

\Rightarrow **Instrumental Variable Method** (操作变数法 or IV 法)

4. Assume the followings:

$$\frac{1}{n}Z'X \longrightarrow M_{zx}, \quad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \quad \frac{1}{n}Z'u \longrightarrow 0$$

5. **Asymptotic Distribution of β_{IV} :**

$$\beta_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u,$$

which is rewritten as:

$$\sqrt{n}(\beta_{IV} - \beta) = \left(\frac{1}{n}Z'X\right)^{-1} \left(\frac{1}{\sqrt{n}}Z'u\right)$$

Applying the Central Limit Theorem to $\left(\frac{1}{\sqrt{n}}Z'u\right)$, we have the following result:

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Therefore,

$$\sqrt{n}(\beta_{IV} - \beta) = \left(\frac{1}{n}Z'X\right)^{-1} \left(\frac{1}{\sqrt{n}}Z'u\right) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}'^{-1})$$

\implies Consistency and Asymptotic Normality

6. The variance of β_{IV} is given by:

$$V(\beta_{IV}) = s^2(Z'X)^{-1}Z'Z(X'Z)^{-1},$$

where

$$s^2 = \frac{(y - X\beta_{IV})'(y - X\beta_{IV})}{n - k}.$$

12.3 Two-Stage Least Squares Method (2 段階最小二乗法, 2SLS or TSLS)

1. Regression Model:

$$y = X\beta + u, \quad u \sim N(0, \sigma^2 I),$$

In the case of $E(X'u) \neq 0$, OLSE is not consistent.

2. Find the variable Z which satisfies $\frac{1}{n}Z'u \rightarrow M_{zu} = 0$.
3. Use $Z = \hat{X}$ for the instrumental variable.

\hat{X} is the predicted value which regresses X on the other exogenous variables, say W .

That is, consider the following regression model:

$$X = WB + V.$$

Estimate B by OLS.

Then, we obtain the prediction:

$$\hat{X} = W\hat{B},$$

where $\hat{B} = (W'W)^{-1}W'X$.

Or, equivalently,

$$\hat{X} = W(W'W)^{-1}W'X.$$

\hat{X} is used for the instrumental variable of X .

4. The IV method is rewritten as:

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'y.$$

Furthermore, β_{IV} is written as follows:

$$\beta_{IV} = \beta + (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'u.$$

Therefore, we obtain the following expression:

$$\begin{aligned}\sqrt{n}(\beta_{IV} - \beta) &= \left(\left(\frac{1}{n} X' W \right) \left(\frac{1}{n} W' W \right)^{-1} \left(\frac{1}{n} X W' \right)' \right)^{-1} \left(\frac{1}{n} X' W \right) \left(\frac{1}{n} W' W \right)^{-1} \left(\frac{1}{\sqrt{n}} W' u \right) \\ &\rightarrow N\left(0, \sigma^2 (M_{xw} M_{ww}^{-1} M'_{xw})^{-1}\right).\end{aligned}$$

5. Clearly, there is no correlation between W and u at least in the limit, i.e.,

$$\text{plim}\left(\frac{1}{n} W' u\right) = 0.$$

6. **Remark:**

$$\hat{X}' X = X' W (W' W)^{-1} W' X = X' W (W' W)^{-1} W' W (W' W)^{-1} W' X = \hat{X}' \hat{X}.$$

Therefore,

$$\beta_{IV} = (\hat{X}' X)^{-1} \hat{X}' y = (\hat{X}' \hat{X})^{-1} \hat{X}' y,$$

which implies the OLS estimator of β in the regression model: $y = \hat{X}\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

Example:

$$y_t = \alpha x_t + \beta z_t + u_t, \quad u_t \sim (0, \sigma^2).$$

Suppose that x_t is correlated with u_t but z_t is not correlated with u_t .

- 1st Step:

Estimate the following regression model:

$$x_t = \gamma w_t + \delta z_t + \cdots + v_t,$$

by OLS. \implies Obtain \hat{x}_t through OLS.

- 2nd Step:

Estimate the following regression model:

$$y_t = \alpha \hat{x}_t + \beta z_t + u_t,$$

by OLS. $\implies \alpha_{iv}$ and β_{iv}

Note as follows. Estimate the following regression model:

$$z_t = \gamma_2 w_t + \delta_2 z_t + \cdots + v_{2t},$$

by OLS.

$\implies \hat{\gamma}_2 = 0, \hat{\delta}_2 = 1$, and the other coefficient estimates are zeros. i.e., $\hat{z}_t = z_t$.

Eviews Command:

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tsls y x z @ w z ...
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