# Econometrics I

## (Thur., 8:50-10:20)

### Room # 4 (法経講義棟)

- The prerequisite of this class is **Basic Statistics (統計基礎)** (by Prof. Fukushige, Tue., 16:20-17:50, this semester) and **Econometrics (エコノメトリックス)** (undergraduate level, next semester, 『計量経済学』山本 拓 著，新世社).
- The class of **Special Lectures in Economics (Statistical Analysis),** 経済学特論（統計解析） (by Prof. Oya, Wed., 10:30-12:00, this semester) should be registered.

# TA Session (by Mr. Yonekura and Mr. Sakamoto):

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| **Tue.,** | **14:40 - 16:10** |
| **Room** | **# 505** (法経大学院総合研究棟) |
| **Content:** | **Basic Statistics, Matrix Algebra, and etc.** |

# 1 Regression Analysis (回帰分析)

## 1.1 Setup of the Model

When $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ are available, suppose that there is a linear relationship between $y$ and $x$, i.e.,

$$y_i = \beta_1 + \beta_2 x_i + u_i, \tag{1}$$

for $i = 1, 2, \cdots, n$. $x_i$ and $y_i$ denote the $i$th observations.

$\longrightarrow$ **Single (or simple) regression model (単回帰モデル)**

$y_i$ is called the **dependent variable (従属変数)** or the **explained variable (被説明変数)**, while $x_i$ is known as the **independent variable (独立変数)** or the **explanatory (or explaining) variable (説明変数)**.

$\beta_1$ = **Intercept (切片),** $\qquad$ $\beta_2$ = **Slope (傾き)**

$\beta_1$ and $\beta_2$ are unknown **parameters (パラメータ，母数)** to be estimated.

$\beta_1$ and $\beta_2$ are called the **regression coefficients (回帰係数)**.

$u_i$ is the unobserved **error term (誤差項)** assumed to be a random variable with mean zero and variance $\sigma^2$.

$\sigma^2$ is also a parameter to be estimated.

$x_i$ is assumed to be **nonstochastic (非確率的)**, but $y_i$ is **stochastic (確率的)** because $y_i$ depends on the error $u_i$.

The error terms $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed, which is called ***iid***. $\longrightarrow$ discussed later.

It is assumed that $u_i$ has a distribution with mean zero, i.e., $\mathrm{E}(u_i) = 0$ is assumed.

Taking the expectation on both sides of (1), the expectation of $y_i$ is represented as:

$$\begin{aligned} \mathrm{E}(y_i) &= \mathrm{E}(\beta_1 + \beta_2 x_i + u_i) = \beta_1 + \beta_2 x_i + \mathrm{E}(u_i) \\ &= \beta_1 + \beta_2 x_i, \end{aligned} \tag{2}$$

for $i = 1, 2, \cdots, n$. Using $\mathrm{E}(y_i)$ we can rewrite (1) as $y_i = \mathrm{E}(y_i) + u_i$.

(2) represents the true regression line.

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be estimates of $\beta_1$ and $\beta_2$.

Replacing $\beta_1$ and $\beta_2$ by $\hat{\beta}_1$ and $\hat{\beta}_2$, (1) turns out to be:

$$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_i + e_i, \tag{3}$$

for $i = 1, 2, \cdots, n$, where $e_i$ is called the **residual (残差)**.

The residual $e_i$ is taken as the experimental value (or realization) of $u_i$.

We define $\hat{y}_i$ as follows:

$$\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 x_i, \tag{4}$$

for $i = 1, 2, \cdots, n$, which is interpreted as the **predicted value (予測値)** of $y_i$.

(4) indicates the estimated regression line, which is different from (2).

Moreover, using $\hat{y}_i$ we can rewrite (3) as $y_i = \hat{y}_i + e_i$.

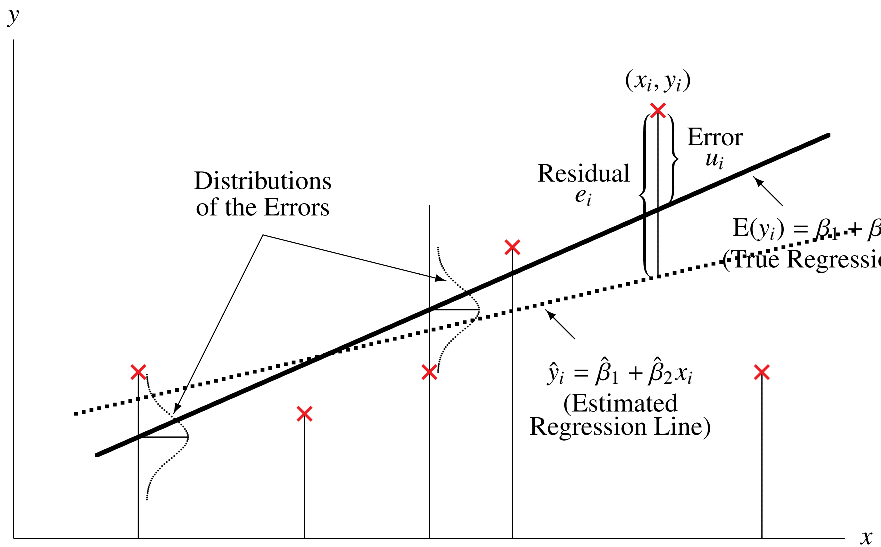
(2) and (4) are displayed in Figure 1.

Consider the case of $n = 6$ for simplicity. $\times$ indicates the observed data series.

The true regression line (2) is represented by the solid line, while the estimated regression line (4) is drawn with the dotted line.

Based on the observed data, $\beta_1$ and $\beta_2$ are estimated as: $\hat{\beta}_1$ and $\hat{\beta}_2$.

**Figure 1. True and Estimated Regression Lines (回帰直線)**




In the next section, we consider how to obtain the estimates of $\beta_1$ and $\beta_2$, i.e., $\hat{\beta}_1$ and $\hat{\beta}_2$.

## 1.2 Ordinary Least Squares Estimation

Suppose that $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ are available.

For the regression model (1), we consider estimating $\beta_1$ and $\beta_2$.

Replacing $\beta_1$ and $\beta_2$ by their estimates $\hat{\beta}_1$ and $\hat{\beta}_2$, remember that the residual $e_i$ is given by:

$$e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i.$$

The sum of squared residuals is defined as follows:

$$S(\hat{\beta}_1, \hat{\beta}_2) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2.$$

It might be plausible to choose the $\hat{\beta}_1$ and $\hat{\beta}_2$ which minimize the sum of squared residuals, i.e., $S(\hat{\beta}_1, \hat{\beta}_2)$.

This method is called the **ordinary least squares estimation (最小二乗法，OLS)**.

To minimize $S(\hat{\beta}_1, \hat{\beta}_2)$ with respect to $\hat{\beta}_1$ and $\hat{\beta}_2$, we set the partial derivatives equal to zero:

$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1} = -2\sum_{i=1}^{n}(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0,$$

$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2} = -2\sum_{i=1}^{n} x_i(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0.$$

The second order condition for minimization is:

$$\begin{pmatrix} \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1^2} & \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1 \partial \hat{\beta}_2} \\ \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2 \partial \hat{\beta}_1} & \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2^2} \end{pmatrix} = \begin{pmatrix} 2n & 2\sum_{i=1}^{n} x_i \\ 2\sum_{i=1}^{n} x_i & 2\sum_{i=1}^{n} x_i^2 \end{pmatrix}$$

should be a positive definite matrix.

The diagonal elements $2n$ and $2\sum_{i=1}^{n} x_i^2$ are positive.

The determinant:

$$\begin{vmatrix} 2n & 2\sum_{i=1}^{n} x_i \\ 2\sum_{i=1}^{n} x_i & 2\sum_{i=1}^{n} x_i^2 \end{vmatrix} = 4n\sum_{i=1}^{n} x_i^2 - 4(\sum_{i=1}^{n} x_i)^2 = 4n\sum_{i=1}^{n}(x_i - \overline{x})^2$$

is positive. $\Longrightarrow$ The second-order condition is satisfied.

The first two equations yield the following two equations:

$$\overline{y} = \hat{\beta}_1 + \hat{\beta}_2 \overline{x}, \tag{5}$$

$$\sum_{i=1}^{n} x_i y_i = n\overline{x}\hat{\beta}_1 + \hat{\beta}_2 \sum_{i=1}^{n} x_i^2, \tag{6}$$

where $\overline{y} = \dfrac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

Multiplying (5) by $n\overline{x}$ and subtracting (6), we can derive $\hat{\beta}_2$ as follows:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{7}$$

From (5), $\hat{\beta}_1$ is directly obtained as follows:

$$\hat{\beta}_1 = \overline{y} - \hat{\beta}_2 \overline{x}. \tag{8}$$

When the observed values are taken for $y_i$ and $x_i$ for $i = 1, 2, \cdots, n$, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimate**s (or simply the **least squares estimate**s, 最小二乗推定値) of $\beta_1$ and $\beta_2$.

When $y_i$ for $i = 1, 2, \cdots, n$ are regarded as the random sample, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimator**s (or the **least squares estimator**s, 最小二乗推定量) of $\beta_1$ and $\beta_2$.

## 1.3 Properties of Least Squares Estimator

Equation (7) is rewritten as:

$$\begin{aligned}\hat{\beta}_2 &= \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})y_i}{\sum_{i=1}^n (x_i - \overline{x})^2} - \frac{\overline{y}\sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2} \\ &= \sum_{i=1}^n \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} y_i = \sum_{i=1}^n \omega_i y_i. \end{aligned} \tag{9}$$

In the third equality, $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$ is utilized because of $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

In the fourth equality, $\omega_i$ is defined as: $\omega_i = \dfrac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$.

$\omega_i$ is nonstochastic because $x_i$ is assumed to be nonstochastic.

$\omega_i$ has the following properties:

$$\sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 0, \tag{10}$$

$$\sum_{i=1}^{n} \omega_i x_i = \sum_{i=1}^{n} \omega_i (x_i - \overline{x}) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 1, \tag{11}$$

$$\sum_{i=1}^{n} \omega_i^2 = \sum_{i=1}^{n} \left( \frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2\right)^2} = \frac{1}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{12}$$

The first equality of (11) comes from (10).

From now on, we focus only on $\hat{\beta}_2$, because usually $\beta_2$ is more important than $\beta_1$ in the regression model (1).

In order to obtain the properties of the least squares estimator $\hat{\beta}_2$, we rewrite (9) as:

$$\begin{aligned}\hat{\beta}_2 &= \sum_{i=1}^{n} \omega_i y_i = \sum_{i=1}^{n} \omega_i (\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i = \beta_2 + \sum_{i=1}^{n} \omega_i u_i. \end{aligned} \tag{13}$$

In the fourth equality of (13), (10) and (11) are utilized.

**[Review] Random Variables**:

Let $X_1$, $X_2$, $\cdots$, $X_n$ be $n$ random variavles, which are mutually independently and identically distributed.

**mutually independent** $\Longrightarrow$ $f(x_i, x_j) = f_i(x_i) f_j(x_j)$ for $i \neq j$.

$f(x_i, x_j)$ denotes a joint distribution of $X_i$ and $X_j$.

$f_i(x)$ indicates a marginal distribution of $X_i$.

**identical** $\Longrightarrow$ $f_i(x) = f_j(x)$ for $i \neq j$.

**[End of Review]**

**[Review] Mean and Variance**:

Let $X$ and $Y$ be random variables (continuous type), which are independently distributed.

**Definition and Formulas:**

- $\mathrm{E}(g(X)) = \displaystyle\int g(x)f(x)\mathrm{d}x$ for a function $g(\cdot)$ and a density function $f(\cdot)$.
- $\mathrm{V}(X) = \mathrm{E}((X-\mu)^2) = \displaystyle\int (x-\mu)^2 f(x)\mathrm{d}x$ for $\mu = \mathrm{E}(X)$.
- $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$ and $\mathrm{V}(aX+b) = \mathrm{V}(aX) = a^2\mathrm{V}(X)$ for constant $a$ and $b$.
- $\mathrm{E}(X \pm Y) = \mathrm{E}(X) \pm \mathrm{E}(Y)$ and $\mathrm{V}(X \pm Y) = \mathrm{V}(X) + \mathrm{V}(Y)$.

**[End of Review]**

**Mean and Variance of $\hat{\beta}_2$:** $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance $\sigma^2$, but they are not necessarily normal.

Remember that we do not need normality assumption to obtain mean and variance but the normality assumption is required to test a hypothesis.

From (13), the expectation of $\hat{\beta}_2$ is derived as follows:

$$\mathrm{E}(\hat{\beta}_2) = \mathrm{E}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \mathrm{E}(\sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) = \beta_2. \tag{14}$$

It is shown from (14) that the ordinary least squares estimator $\hat{\beta}_2$ is an unbiased estimator of $\beta_2$.

From (13), the variance of $\hat{\beta}_2$ is computed as:

$$\mathrm{V}(\hat{\beta}_2) = \mathrm{V}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \mathrm{V}(\sum_{i=1}^{n} \omega_i u_i) = \sum_{i=1}^{n} \mathrm{V}(\omega_i u_i) = \sum_{i=1}^{n} \omega_i^2 \mathrm{V}(u_i)$$

$$= \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{15}$$

The third equality holds because $u_1, u_2, \cdots, u_n$ are mutually independent.

The last equality comes from (12).

Thus, $\mathrm{E}(\hat{\beta}_2)$ and $\mathrm{V}(\hat{\beta}_2)$ are given by (14) and (15).

**[Review] Three Good Properties on Estimator:**

$\theta$ : Parameter

$\hat{\theta}$ : Estimator of $\theta$, i.e., $\hat{\theta} = \hat{\theta}(X_1, X_2, \cdots, X_n)$,

where $X_1, X_2, \cdots, X_n$ are mutually independent random variables.

(*) Estimate of $\theta$: $\hat{\theta} = \hat{\theta}(x_1, x_2, \cdots, x_n)$, where $x_i$ denotes the observed data of $X_i$.

- **Unbiasedness (不偏性)**: $\mathrm{E}(\hat{\theta}) = \theta$.
- **Efficiency (有効性)**:

  The minimum variance estimator within all the unbiased estimators.

(*) It is not easy to check efficiency in general. Instead, consider the **best linear unbiased estimator (BLUE, 最良線型不偏推定量)**.

- **Consistency (一致性)**: $\hat{\theta} \longrightarrow \theta$ as $n \longrightarrow \infty$. Note that $\hat{\theta}$ depends on # of obs.

**[End of Review]**

**Gauss-Markov Theorem (ガウス・マルコフ定理):** It has been discussed above that $\hat{\beta}_2$ is represented as (9), which implies that $\hat{\beta}_2$ is a linear estimator, i.e., linear in $y_i$.

In addition, (14) indicates that $\hat{\beta}_2$ is an unbiased estimator.

Therefore, summarizing these two facts, it is shown that $\hat{\beta}_2$ is a **linear unbiased estimator (線形不偏推定量)**.

Furthermore, here we show that $\hat{\beta}_2$ has minimum variance within a class of the linear unbiased estimators.

Consider the alternative linear unbiased estimator $\tilde{\beta}_2$ as follows:

$$\tilde{\beta}_2 = \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i) y_i,$$

where $c_i = \omega_i + d_i$ is defined and $d_i$ is nonstochastic.

Then, $\tilde{\beta}_2$ is transformed into:

$$\begin{aligned}
\tilde{\beta}_2 &= \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i)(\beta_1 + \beta_2 x_i + u_i) \\
&= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} d_i u_i \\
&= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i u_i + \sum_{i=1}^{n} d_i u_i.
\end{aligned}$$

Equations (10) and (11) are used in the forth equality.

Taking the expectation on both sides of the above equation, we obtain:

$$\begin{aligned}
\mathrm{E}(\tilde{\beta}_2) &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) + \sum_{i=1}^{n} d_i \mathrm{E}(u_i) \\
&= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i.
\end{aligned}$$

Note that $d_i$ is not a random variable and that $\mathrm{E}(u_i) = 0$.

Since $\tilde{\beta}_2$ is assumed to be unbiased, we need the following conditions:

$$\sum_{i=1}^{n} d_i = 0, \qquad \sum_{i=1}^{n} d_i x_i = 0.$$

When these conditions hold, we can rewrite $\tilde{\beta}_2$ as:

$$\tilde{\beta}_2 = \beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i.$$

The variance of $\tilde{\beta}_2$ is derived as:

$$\begin{aligned}
\mathrm{V}(\tilde{\beta}_2) &= \mathrm{V}\Big(\beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = \mathrm{V}\Big(\sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = \sum_{i=1}^{n} \mathrm{V}\Big((\omega_i + d_i) u_i\Big) \\
&= \sum_{i=1}^{n} (\omega_i + d_i)^2 \mathrm{V}(u_i) = \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + 2 \sum_{i=1}^{n} \omega_i d_i + \sum_{i=1}^{n} d_i^2) \\
&= \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + \sum_{i=1}^{n} d_i^2).
\end{aligned}$$

From unbiasedness of $\tilde{\beta}_2$, using $\sum_{i=1}^n d_i = 0$ and $\sum_{i=1}^n d_i x_i = 0$, we obtain:

$$\sum_{i=1}^n \omega_i d_i = \frac{\sum_{i=1}^n (x_i - \overline{x}) d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n x_i d_i - \overline{x} \sum_{i=1}^n d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = 0,$$

which is utilized to obtain the variance of $\tilde{\beta}_2$ in the third line of the above equation.

From (15), the variance of $\hat{\beta}_2$ is given by: $\mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^n \omega_i^2$.

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}_2) \geq \mathrm{V}(\hat{\beta}_2),$$

because of $\sum_{i=1}^n d_i^2 \geq 0$.

When $\sum_{i=1}^n d_i^2 = 0$, i.e., when $d_1 = d_2 = \cdots = d_n = 0$,

we have the equality: $\mathrm{V}(\tilde{\beta}_2) = \mathrm{V}(\hat{\beta}_2)$.

Thus, in the case of $d_1 = d_2 = \cdots = d_n = 0$, $\hat{\beta}_2$ is equivalent to $\tilde{\beta}_2$.

As shown above, the least squares estimator $\hat{\beta}_2$ gives us the **minimum variance linear unbiased estimator (最小分散線形不偏推定量)**, or equivalently the **best linear unbiased estimator (最良線形不偏推定量，BLUE)**, which is called the **Gauss-Markov theorem (ガウス・マルコフ定理)**.

**Asymptotic Properties (漸近的性質) of $\hat{\beta}_2$:** We assume that as $n$ goes to infinity we have the following:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2 \longrightarrow m<\infty,$$

where $m$ is a constant value. From (12), we obtain:

$$n\sum_{i=1}^{n}\omega_i^2=\frac{1}{(1/n)\sum_{i=1}^{n}(x_i-\overline{x})} \longrightarrow \frac{1}{m}.$$

Note that $f(x_n)\longrightarrow f(m)$ when $x_n\longrightarrow m$, called **Slutsky's theorem (スルツキー定理)**, where $m$ is a constant value and $f(\cdot)$ is a function.

We show both **consistency** (一致性) of $\hat{\beta}_2$ and **asymptotic normality** (漸近正規性) of $\sqrt{n}(\hat{\beta}_2-\beta_2)$.

● First, we prove that $\hat{\beta}_2$ is a consistent estimator of $\beta_2$.

**[Review] Chebyshev's inequality (チェビシェフの不等式)** is given by:

$$P(|X-\mu|>\epsilon) \le \frac{\sigma^2}{\epsilon^2}, \qquad \text{where } \mu = \mathrm{E}(X),\ \sigma^2 = \mathrm{V}(X) \text{ and any } \epsilon > 0.$$

**[End of Review]**

Replace $X$, $\mathrm{E}(X)$ and $\mathrm{V}(X)$ by:

$$\hat{\beta}_2, \qquad \mathrm{E}(\hat{\beta}_2) = \beta_2, \quad \text{and} \quad \mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})}.$$

Then, when $n \longrightarrow \infty$, we obtain the following result:

$$P(|\hat{\beta}_2 - \beta_2| > \epsilon) \le \frac{\sigma^2 \sum_{i=1}^{n} \omega_i^2}{\epsilon^2} = \frac{\sigma^2 n \sum_{i=1}^{n} \omega_i^2}{n\epsilon^2} \longrightarrow 0,$$

where $\sum_{i=1}^{n} \omega_i^2 \longrightarrow 0$ because $n \sum_{i=1}^{n} \omega_i^2 \longrightarrow \dfrac{1}{m}$ from the assumption.

Thus, we obtain the result that $\hat{\beta}_2 \longrightarrow \beta_2$ as $n \longrightarrow \infty$.

Therefore, we can conclude that $\hat{\beta}_2$ is a **consistent estimator (一致推定量)** of $\beta_2$.

● Next, we want to show that $\sqrt{n}(\hat{\beta}_2 - \beta_2)$ is asymptotically normal.

**[Review]** The **Central Limit Theorem (中心極限定理, CLT)** is: for random variables $X_1, X_2, \cdots, X_n$,

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\sum_{i=1}^n X_i - \mathrm{E}(\sum_{i=1}^n X_i)}{\sqrt{\mathrm{V}(\sum_{i=1}^n X_i)}} \longrightarrow N(0, 1), \qquad \text{as} \qquad n \longrightarrow \infty,$$

where $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n X_i$.

$X_1, X_2, \cdots, X_n$ are not necesarily iid, if $\mathrm{V}(\overline{X})$ is finite as $n$ goes to infinity.

**[End of Review]**

Note that $\hat{\beta}_2 = \beta_2 + \sum_{i=1}^n \omega_i u_i$ as in (13), and $X_i$ is replaced by $\omega_i u_i$.

From the central limit theorem, asymptotic normality is shown as follows:

$$\frac{\sum_{i=1}^n \omega_i u_i - \mathrm{E}(\sum_{i=1}^n \omega_i u_i)}{\sqrt{\mathrm{V}(\sum_{i=1}^n \omega_i u_i)}} = \frac{\sum_{i=1}^n \omega_i u_i}{\sigma \sqrt{\sum_{i=1}^n \omega_i^2}} = \frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0, 1),$$

where

- $\mathrm{E}(\sum_{i=1}^n \omega_i u_i) = 0$,
- $\mathrm{V}(\sum_{i=1}^n \omega_i u_i) = \sigma^2 \sum_{i=1}^n \omega_i^2$, and
- $\sum_{i=1}^n \omega_i u_i = \hat{\beta}_2 - \beta_2$

are substituted in the first and second equalities.

Moreover, we can rewrite as follows:

$$\frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} = \frac{\sqrt{n}(\hat{\beta}_2 - \beta_2)}{\sigma / \sqrt{(1/n) \sum_{i=1}^n (x_i - \overline{x})^2}}.$$

Replacing $(1/n) \sum_{i=1}^n (x_i - \overline{x})^2$ by its converged value $m$, we have:

$$\frac{\sqrt{n}(\hat{\beta}_2 - \beta_2)}{\sigma / \sqrt{m}} \longrightarrow N(0, 1),$$

which implies

$$\sqrt{n}(\hat{\beta}_2 - \beta_2) \longrightarrow N(0, \frac{\sigma^2}{m}).$$

Thus, the asymptotic normality of $\sqrt{n}(\hat{\beta}_2 - \beta_2)$ is shown.

Finally, replacing $\sigma^2$ by its consistent estimator $s^2$, it is known as follows:

$$\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \longrightarrow N(0, 1), \tag{16}$$

where $s^2$ is defined as:

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n} e_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2, \tag{17}$$

which is a consistent and unbiased estimator of $\sigma^2$. $\longrightarrow$ Proved later.

Thus, using (16), in large sample we can construct the confidence interval and test the hypothesis.

**[Review] Confidence Interval (信頼区間，区間推定)):**

Suppose $X_1, X_2, \cdots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$. $\longrightarrow$ No N assumption

From CLT, $\dfrac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \longrightarrow N(0, 1)$.

Replacing $\sigma^2$ by $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, we have: $\dfrac{\overline{X} - \mu}{S / \sqrt{n}} \longrightarrow N(0, 1)$.

That is, for large $n$,

$$P\Big(-1.96 < \frac{\overline{X} - \mu}{S / \sqrt{n}} < 1.96\Big) = 0.95, \text{ i.e., } P\Big(\overline{X} - 1.96\frac{S}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{S}{\sqrt{n}}\Big) = 0.95.$$

Note that 1.96 is obtained from the normal distribution table.

Then, replacing the estimators $\overline{X}$ and $S^2$ by the estimates $\overline{x}$ and $s^2$, we obtain the 95% confidence interval of $\mu$ as follows:

$$(\overline{x} - 1.96\frac{s}{\sqrt{n}},\ \overline{x} + 1.96\frac{s}{\sqrt{n}}).$$

**[End of Review]**

Going back to OLS, we have:

$$\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0,1).$$

Therefore,

$$P\Big(-2.576 < \frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} < 2.576\Big) = 0.99,$$

i.e.,

$$P\Big(\hat{\beta}_2 - 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} < \beta_2 < \hat{\beta}_2 + 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}\Big) = 0.99.$$

Note that 2.576 is 0.005 value of $N(0,1)$, which comes from the statistical table.

Thus, the 99% confidence interval of $\beta_2$ is:

$$\Big(\hat{\beta}_2 - 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}},\ \hat{\beta}_2 + 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}\Big),$$

where $\hat{\beta}_2$ and $s^2$ should be replaced by the observed data.

**[Review] Testing the Hypothesis (仮説検定):**

Suppose that $X_1, X_2, \cdots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$.

From CLT, $\dfrac{\overline{X} - \mu}{S/\sqrt{n}} \longrightarrow N(0,1)$, where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, which is known as the unbiased estimator of $\sigma^2$.

- The null hypothesis $H_0: \ \mu = \mu_0$, where $\mu_0$ is a fixed number.
- The alternative hypothesis $H_1: \ \mu \neq \mu_0$

Under the null hypothesis, in large sample we have the following disribution:

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim N(0,1).$$

Replacing $\overline{X}$ and $S^2$ by $\overline{x}$ and $s^2$, compare $\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$ and $N(0,1)$.

$H_0$ is rejected at significance level 0.05 when $\left|\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}\right| > 1.96$.

**[End of Review]**

In the case of OLS, the hypotheses are as follows:

- The null hypothesis $H_0 : \beta_2 = \beta_2^*$
- The alternative hypothesis $H_1 : \beta_2 \neq \beta_2^*$

Under $H_0$, in large sample,

$$\frac{\hat{\beta}_2 - \beta_2^*}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \sim N(0, 1).$$

Replacing $\hat{\beta}_2$ and $s^2$ by the observed data, compare $\dfrac{\hat{\beta}_2 - \beta_2^*}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}$ and $N(0, 1)$.

$H_0$ is rejected at significance level 0.05 when $\left|\dfrac{\hat{\beta}_2 - \beta_2^*}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}\right| > 1.96$.

**Exact Distribution of $\hat{\beta}_2$:** We have shown asymptotic normality of $\sqrt{n}(\hat{\beta}_2 - \beta_2)$, which is one of the large sample properties.

Now, we discuss the small sample properties of $\hat{\beta}_2$.

In order to obtain the distribution of $\hat{\beta}_2$ in small sample, the distribution of the error term has to be assumed.

Therefore, the extra assumption is that $u_i \sim N(0, \sigma^2)$.

Writing (13), again, $\hat{\beta}_2$ is represented as:

$$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} \omega_i u_i.$$

First, we obtain the distribution of the second term in the above equation.

**[Review] Content of Special Lectures in Economics (Statistical Analysis)**

Note that the **moment-generating function (積率母関数, MGF)** is given by $M(\theta) \equiv \mathrm{E}(\exp(\theta X)) = \exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$ when $X \sim N(\mu, \sigma^2)$.

$X_1$, $X_2$, $\cdots$, $X_n$ are mutually independently distributed as $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \cdots, n$.

MGF of $X_i$ is $M_i(\theta) \equiv \mathrm{E}(\exp(\theta X_i)) = \exp(\mu_i\theta + \frac{1}{2}\sigma_i^2\theta^2)$.

Consider the distribution of $Y = \sum_{i=1}^n (a_i + b_i X_i)$, where $a_i$ and $b_i$ are constant.

$$
\begin{aligned}
M_y(\theta) &\equiv \mathrm{E}(\exp(\theta Y)) = \mathrm{E}(\exp(\theta \textstyle\sum_{i=1}^n (a_i + b_i X_i))) \\
&= \textstyle\prod_{i=1}^n \exp(\theta a_i)\mathrm{E}(\exp(\theta b_i X_i)) = \prod_{i=1}^n \exp(\theta a_i) M_i(\theta b_i) \\
&= \textstyle\prod_{i=1}^n \exp(\theta a_i) \exp(\mu_i\theta b_i + \frac{1}{2}\sigma_i^2(\theta b_i)^2) = \exp(\theta \sum_{i=1}^n (a_i + b_i\mu_i) + \frac{1}{2}\theta^2 \sum_{i=1}^n b_i^2\sigma_i^2),
\end{aligned}
$$

which implies that $Y \sim N(\sum_{i=1}^n (a_i + b_i\mu_i), \sum_{i=1}^n b_i^2\sigma_i^2)$.

**[End of Review]**

Substitute $a_i = 0$, $\mu_i = 0$, $b_i = \omega_i$ and $\sigma_i^2 = \sigma^2$.

Then, using the moment-generating function, $\sum_{i=1}^n \omega_i u_i$ is distributed as:

$$\sum_{i=1}^n \omega_i u_i \sim N(0,\ \sigma^2 \sum_{i=1}^n \omega_i^2).$$

Therefore, $\hat{\beta}_2$ is distributed as:

$$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^n \omega_i u_i \sim N(\beta_2,\ \sigma^2 \sum_{i=1}^n \omega_i^2),$$

or equivalently,

$$\frac{\hat{\beta}_2 - \beta_2}{\sigma \sqrt{\sum_{i=1}^n \omega_i^2}} = \frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \sim N(0, 1),$$

for any $n$.

**[Review 1]** $\quad t$ **Distribution:**

$Z \sim N(0,1)$, $V \sim \chi^2(k)$, and $Z$ is independent of $V$. $\quad$ Then, $\dfrac{Z}{\sqrt{V/k}} \sim t(k)$.

**[End of Review 1]**

**[Review 2]** $\quad t$ **Distribution:**

Suppose that $X_1, X_2. \cdots, X_n$ are mutually independently, identically and normally distributed with mean $\mu$ and variance $\sigma^2$.

$\overline{X} \sim N(\mu, \sigma^2/n)$, i.e., $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$.

Define $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, which is an unbiased estimator of $\sigma^2$.

It is known that $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ and $\overline{X}$ is independesnt of $S^2$. (The proof is skipped.)

Then, we obtain $\dfrac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2}\Big/(n-1)}} = \dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1).$

As a result, replacing $\sigma^2$ by $S^2$, $\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1).$

**[End of Review 2]**

Back to OLS:

Replacing $\sigma^2$ by its estimator $s^2$ defined in (17), it is known that we have:

$$\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \sim t(n-2),$$

where $t(n-2)$ denotes $t$ distribution with $n-2$ degrees of freedom.

Thus, under normality assumption on the error term $u_i$, the $t(n-2)$ distribution is used for the confidence interval and the testing hypothesis in small sample.

Or, taking the square on both sides,

$$\Big(\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}\Big)^2 \sim F(1, n-2),$$

which will be proved later.

Before going to **multiple regression model (重回帰モデル)**,

# 2 Some Formulas of Matrix Algebra

1. Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lk} \end{pmatrix} = [a_{ij}]$,

   which is a $l \times k$ matrix, where $a_{ij}$ denotes $i$th row and $j$th column of $A$.

   The **transposed matrix (転置行列)** of $A$, denoted by $A'$, is defined as:

   $$A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{l1} \\ a_{12} & a_{22} & \cdots & a_{l2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{lk} \end{pmatrix} = [a_{ji}],$$

   where the $i$th row of $A'$ is the $i$th column of $A$.

2. $(Ax)' = x'A'$,

where $A$ and $x$ are a $l \times k$ matrix and a $k \times 1$ vector, respectively.

3. $a' = a$,

where $a$ denotes a scalar.

4. $\dfrac{\partial a'x}{\partial x} = a$,

where $a$ and $x$ are $k \times 1$ vectors.

5. $\dfrac{\partial x'Ax}{\partial x} = (A + A')x$,

where $A$ and $x$ are a $k \times k$ matrix and a $k \times 1$ vector, respectively.

Especially, when $A$ is symmetric,

$$\frac{\partial x'Ax}{\partial x} = 2Ax.$$

6. Let $A$ and $B$ be $k \times k$ matrices, and $I_k$ be a $k \times k$ **identity matrix (単位行列)** (one in the diagonal elements and zero in the other elements).

   When $AB = I_k$, $B$ is called the **inverse matrix (逆行列)** of $A$, denoted by $B = A^{-1}$.

   That is, $AA^{-1} = A^{-1}A = I_k$.

7. Let $A$ be a $k \times k$ matrix and $x$ be a $k \times 1$ vector.

   If $A$ is a **positive definite matrix (正値定符号行列)**, for any $x$ except for $x = 0$ we have:

$$x'Ax > 0.$$

   If $A$ is a **positive semidefinite matrix (非負値定符号行列)**, for any $x$ except for $x = 0$ we have:

$$x'Ax \geq 0.$$

If $A$ is a **negative definite matrix (負値定符号行列)**, for any $x$ except for $x = 0$ we have:

$$x'Ax < 0.$$

If $A$ is a **negative semidefinite matrix (非正値定符号行列)**, for any $x$ except for $x = 0$ we have:

$$x'Ax \le 0.$$

**Trace, Rank and etc.:** $A : k \times k, \qquad B : n \times k, \qquad C : k \times n.$

1. The **trace (トレース)** of $A$ is: $\mathrm{tr}(A) = \sum_{i=1}^{k} a_{ii}$, where $A = [a_{ij}]$ .

2. The **rank (ランク，階数)** of $A$ is the maximum number of linearly independent column (or row) vectors of $A$, which is denoted by $\mathrm{rank}(A)$.

3. If $A$ is an **idempotent matrix** (べき等行列), $A = A^2$ .

4. If $A$ is an idempotent and symmetric matrix, $A = A^2 = A'A$ .

5. $A$ is idempotent if and only if the eigen values of $A$ consist of 1 and 0.

6. If $A$ is idempotent, rank($A$) =tr($A$) .

7. tr($BC$) =tr($CB$)

**Distributions in Matrix Form:**

1. Let $X$, $\mu$ and $\Sigma$ be $k \times 1$, $k \times 1$ and $k \times k$ matrices.

   When $X \sim N(\mu, \Sigma)$, the density function of $X$ is given by:

$$f(x) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big).$$

$\mathrm{E}(X) = \mu$ and $\mathrm{V}(X) = \mathrm{E}\Big((X-\mu)(X-\mu)'\Big) = \Sigma$

The moment-generating function: $\phi(\theta) = \mathrm{E}\Big(\exp(\theta' X)\Big) = \exp(\theta'\mu + \frac{1}{2}\theta'\Sigma\theta)$

(*) In the univariate case, when $X \sim N(\mu, \sigma^2)$, the density function of $X$ is:

$$f(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

2. If $X \sim N(\mu, \Sigma)$, then $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2(k)$.

   Note that $X'X \sim \chi^2(k)$ when $X \sim N(0, I_k)$.

3. $X$: $n \times 1$, $Y$: $m \times 1$, $X \sim N(\mu_x, \Sigma_x)$, $Y \sim N(\mu_y, \Sigma_y)$

   $X$ is independent of $Y$, i.e., $\mathrm{E}\Big((X-\mu_x)(Y-\mu_y)'\Big) = 0$ in the case of normal random variables.

$$\frac{(X-\mu_x)'\Sigma_x^{-1}(X-\mu_x)/n}{(Y-\mu_y)'\Sigma_y^{-1}(Y-\mu_y)/m} \sim F(n, m)$$

4. If $X \sim N(0, \sigma^2 I_n)$ and $A$ is a symmetric idempotent $n \times n$ matrix of rank $G$, then $X'AX/\sigma^2 \sim \chi^2(G)$.

   Note that $X'AX = (AX)'(AX)$ and $\text{rank}(A) = \text{tr}(A)$ because $A$ is idempotent.

5. If $X \sim N(0, \sigma^2 I_n)$, $A$ and $B$ are symmetric idempotent $n \times n$ matrices of rank $G$ and $K$, and $AB = 0$, then

$$\frac{X'AX}{G\sigma^2} \Big/ \frac{X'BX}{K\sigma^2} = \frac{X'AX/G}{X'BX/K} \sim F(G, K).$$

# 3 Multiple Regression Model (重回帰モデル)

Up to now, only one independent variable, i.e., $x_i$, is taken into the regression model.

We extend it to more independent variables, which is called the **multiple regression model (重回帰モデル)**.

We consider the following regression model:

$$y_i = \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_k x_{i,k} + u_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,k}) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + u_i = x_i \beta + u_i,$$

for $i = 1, 2, \cdots, n$, where $x_i$ and $\beta$ denote a $1 \times k$ vector of the independent variables and a $k \times 1$ vector of the unknown parameters to be estimated, which are given by:

$$x_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,k}), \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}.$$

$x_{i,j}$ denotes the $i$th observation of the $j$th independent variable.

The case of $k = 2$ and $x_{i,1} = 1$ for all $i$ is exactly equivalent to (1).

Therefore, the matrix form above is a generalization of (1).

Writing all the equations for $i = 1, 2, \cdots, n$, we have:

$$\begin{aligned} y_1 &= \beta_1 x_{1,1} + \beta_2 x_{1,2} + \cdots + \beta_k x_{1,k} + u_1 = x_1\beta + u_1, \\ y_2 &= \beta_1 x_{2,1} + \beta_2 x_{2,2} + \cdots + \beta_k x_{2,k} + u_2 = x_2\beta + u_2, \\ &\vdots \\ y_n &= \beta_1 x_{n,1} + \beta_2 x_{n,2} + \cdots + \beta_k x_{n,k} + u_n = x_n\beta + u_n, \end{aligned}$$

which is rewritten as:

$$
\begin{aligned}
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} &= \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \beta + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.
\end{aligned}
$$

Again, the above equation is compactly rewritten as:

$$
y = X\beta + u, \tag{18}
$$

where $y$, $X$ and $u$ are denoted by:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Utilizing the matrix form (18), we derive the ordinary least squares estimator of $\beta$, denoted by $\hat{\beta}$.

In (18), replacing $\beta$ by $\hat{\beta}$, we have the following equation:

$$y = X\hat{\beta} + e,$$

where $e$ denotes a $n \times 1$ vector of the residuals.

The $i$th element of $e$ is given by $e_i$.

The sum of squared residuals is written as follows:

$$S(\hat{\beta}) = \sum_{i=1}^{n} e_i^2 = e'e = (y - X\hat{\beta})'(y - X\hat{\beta}) = (y' - \hat{\beta}'X')(y - X\hat{\beta})$$
$$= y'y - y'X\hat{\beta} - \hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} = y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}.$$

In the last equality, note that $\hat{\beta}'X'y = y'X\hat{\beta}$ because both are scalars.

To minimize $S(\hat{\beta})$ with respect to $\hat{\beta}$, we set the first derivative of $S(\hat{\beta})$ equal to zero, i.e.,

$$\frac{\partial S(\hat{\beta})}{\partial \hat{\beta}} = -2X'y + 2X'X\hat{\beta} = 0.$$

Solving the equation above with respect to $\hat{\beta}$, the **ordinary least squares estimator (OLS,** 最小自乗推定量) of $\beta$ is given by:

$$\hat{\beta} = (X'X)^{-1}X'y. \tag{19}$$

Thus, the ordinary least squares estimator is derived in the matrix form.

(*) Remark

The second order condition for minimization:

$$\frac{\partial^2 S(\hat{\beta})}{\partial\hat{\beta}\partial\hat{\beta}'} = 2X'X$$

is a positive definite matrix.

Set $c = Xd$.

For any $d \neq 0$, we have $c'c = d'X'Xd > 0$.

Now, in order to obtain the properties of $\hat{\beta}$ such as mean, variance, distribution and so on, (19) is rewritten as follows:

$$\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u$$
$$= \beta + (X'X)^{-1}X'u. \tag{20}$$

Taking the expectation on both sides of (20), we have the following:

$$\mathrm{E}(\hat{\beta}) = \mathrm{E}(\beta + (X'X)^{-1}X'u) = \beta + (X'X)^{-1}X'\mathrm{E}(u) = \beta,$$

because of $\mathrm{E}(u) = 0$ by the assumption of the error term $u_i$.

Thus, unbiasedness of $\hat{\beta}$ is shown.

The variance of $\hat{\beta}$ is obtained as:

$$\begin{aligned}
\mathrm{V}(\hat{\beta}) &= \mathrm{E}((\hat{\beta}-\beta)(\hat{\beta}-\beta)') = \mathrm{E}\Big((X'X)^{-1}X'u((X'X)^{-1}X'u)'\Big) \\
&= \mathrm{E}((X'X)^{-1}X'uu'X(X'X)^{-1}) = (X'X)^{-1}X'\mathrm{E}(uu')X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.
\end{aligned}$$

The first equality is the definition of variance in the case of vector.

In the fifth equality, $\mathrm{E}(uu') = \sigma^2 I_n$ is used, which implies that $\mathrm{E}(u_i^2) = \sigma^2$ for all $i$ and $\mathrm{E}(u_i u_j) = 0$ for $i \neq j$.

Remember that $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance $\sigma^2$.

Under normality assumption on the error term $u$, it is known that the distribution of $\hat{\beta}$ is given by:

$$\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1}).$$

**Proof:**

First, when $X \sim N(\mu, \Sigma)$, the moment-generating function, i.e., $\phi(\theta)$, is given by:

$$\phi(\theta) \equiv \mathrm{E}\Big(\exp(\theta' X)\Big) = \exp\Big(\theta'\mu + \frac{1}{2}\theta'\Sigma\theta\Big)$$

$\theta_u$: $n \times 1$, $\qquad u$: $n \times 1$, $\qquad \theta_\beta$: $k \times 1$, $\qquad \hat{\beta}$: $k \times 1$

The moment-generating function of $u$, i.e., $\phi_u(\theta_u)$, is:

$$\phi_u(\theta_u) \equiv \mathrm{E}\Big(\exp(\theta_u' u)\Big) = \exp\Big(\frac{\sigma^2}{2}\theta_u'\theta_u\Big),$$

which is $N(0, \sigma^2 I_n)$.

The moment-generating function of $\hat{\beta}$, i.e., $\phi_\beta(\theta_\beta)$, is:

$$\begin{aligned}
\phi_\beta(\theta_\beta) &\equiv \mathrm{E}\Big(\exp(\theta_\beta'\hat{\beta})\Big) = \mathrm{E}\Big(\exp(\theta_\beta'\beta + \theta_\beta'(X'X)^{-1}X'u)\Big) \\
&= \exp(\theta_\beta'\beta)\mathrm{E}\Big(\exp(\theta_\beta'(X'X)^{-1}X'u)\Big) = \exp(\theta_\beta'\beta)\phi_u\Big(\theta_\beta'(X'X)^{-1}X'\Big) \\
&= \exp(\theta_\beta'\beta)\exp\Big(\frac{\sigma^2}{2}\theta_\beta'(X'X)^{-1}\theta_\beta\Big) = \exp\Big(\theta_\beta'\beta + \frac{\sigma^2}{2}\theta_\beta'(X'X)^{-1}\theta_\beta\Big),
\end{aligned}$$

which is equivalent to the normal distribution with mean $\beta$ and variance $\sigma^2(X'X)^{-1}$.

Note that $\theta_u = X(X'X)^{-1}\theta_\beta$. QED

Taking the $j$th element of $\hat{\beta}$, its distribution is given by:

$$\hat{\beta}_j \sim N(\beta_j, \sigma^2 a_{jj}), \qquad \text{i.e.,} \qquad \frac{\hat{\beta}_j - \beta_j}{\sigma \sqrt{a_{jj}}} \sim N(0, 1),$$

where $a_{jj}$ denotes the $j$th diagonal element of $(X'X)^{-1}$.

Replacing $\sigma^2$ by its estimator $s^2$, we have the following $t$ distribution:

$$\frac{\hat{\beta}_j - \beta_j}{s \sqrt{a_{jj}}} \sim t(n - k),$$

where $t(n - k)$ denotes the $t$ distribution with $n - k$ degrees of freedom.

**[Review] Trace (トレース):**

1. $A$: $n \times n$, $\quad \mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$, where $a_{ij}$ denotes an element in the $i$th row and the $j$th column of a matrix $A$.

2. $a$: scalar $(1 \times 1)$, $\quad \mathrm{tr}(a) = a$

3. $A$: $n \times k$, $B$: $k \times n$, $\quad \mathrm{tr}(AB) = \mathrm{tr}(BA)$

4. $\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$

5. When $X$ is a square matrix of random variables, $\mathrm{E}(\mathrm{tr}(AX)) = \mathrm{tr}(A\mathrm{E}(X))$

**End of Review**

$s^2$ is taken as follows:

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} e_i^2 = \frac{1}{n-k}e'e = \frac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta}),$$

which leads to an unbiased estimator of $\sigma^2$.

**Proof:**

Substitute $y = X\beta + u$ and $\hat{\beta} = \beta + (X'X)^{-1}X'u$ into $e = y - X\hat{\beta}$.

$$\begin{aligned} e = y - X\hat{\beta} &= X\beta + u - X(\beta + (X'X)^{-1}X'u) \\ &= u - X(X'X)^{-1}X'u = (I_n - X(X'X)^{-1}X')u \end{aligned}$$

$I_n - X(X'X)^{-1}X'$ is idempotent and symmetric, because we have:

$$(I_n - X(X'X)^{-1}X')(I_n - X(X'X)^{-1}X') = I_n - X(X'X)^{-1}X,'$$
$$(I_n - X(X'X)^{-1}X')' = I_n - X(X'X)^{-1}X'.$$

$s^2$ is rewritten as follows:

$$
\begin{aligned}
s^2 &= \frac{1}{n-k}e'e = \frac{1}{n-k}((I_n - X(X'X)^{-1}X')u)'(I_n - X(X'X)^{-1}X')u \\
&= \frac{1}{n-k}u'(I_n - X(X'X)^{-1}X')'(I_n - X(X'X)^{-1}X')u \\
&= \frac{1}{n-k}u'(I_n - X(X'X)^{-1}X')u
\end{aligned}
$$

Take the expectation of $u'(I_n - X(X'X)^{-1}X')u$ and note that $\mathrm{tr}(a) = a$ for a scalar $a$.

$$
\begin{aligned}
\mathrm{E}(s^2) &= \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\Big(u'(I_n - X(X'X)^{-1}X')u\Big)\Big) = \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')uu'\Big)\Big) \\
&= \frac{1}{n-k}\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')\mathrm{E}(uu')\Big) = \frac{1}{n-k}\sigma^2\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')I_n\Big) \\
&= \frac{1}{n-k}\sigma^2\mathrm{tr}(I_n - X(X'X)^{-1}X') = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(X(X'X)^{-1}X')) \\
&= \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}((X'X)^{-1}X'X)) = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(I_k)) \\
&= \frac{1}{n-k}\sigma^2(n-k) = \sigma^2
\end{aligned}
$$

$\longrightarrow$ $s^2$ is an unbiased estimator of $\sigma^2$.

Note that we do not need normality assumption for unbiasedness of $s^2$.

**[Review]**

- $X'X \sim \chi^2(n)$ for $X \sim N(0, I_n)$.

- $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2(n)$ for $X \sim N(\mu, \Sigma)$.

- $\dfrac{X'X}{\sigma^2} \sim \chi^2(n)$ for $X \sim N(0, \sigma^2 I_n)$.

- $\dfrac{X'AX}{\sigma^2} \sim \chi^2(G)$, where $X \sim N(0, \sigma^2 I_n)$ and $A$ is a symmetric idempotent $n \times n$ matrix of rank $G \le n$.

Remember that $G = \text{Rank}(A) = \text{tr}(A)$ when $A$ is symmetric and idempotent.

**[End of Review]**

Under normality assumption for $u$, the distribution of $s^2$ is:

$$\frac{(n-k)s^2}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(\mathrm{tr}(I_n - X(X'X)^{-1}X'))$$

Note that $\quad \mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k$, because

$$\mathrm{tr}(I_n) = n$$

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

**Asymptotic Normality (without normality assumption on $u$):** Using the central limit theorem, without normality assumption we can show that as $n \longrightarrow \infty$, under the condition of $\frac{1}{n}X'X \longrightarrow M$ we have the following result:

$$\frac{\hat{\beta}_j - \beta_j}{s\sqrt{a_{jj}}} \longrightarrow N(0,1),$$

where $M$ denotes a $k \times k$ constant matrix.

Thus, we can construct the confidence interval and the testing procedure, using the $t$ distribution under the normality assumption or the normal distribution without the normality assumption.

# 4 Properties of OLSE

1. Properties of $\hat{\beta}$ : **BLUE (best linear unbiased estimator，最良線形不偏推定量)**, i.e., minimum variance within the class of linear unbiased estimators **(Gauss-Markov theorem，ガウス・マルコフの定理)**

   **Proof:**

   Consider another linear unbiased estimator, which is denoted by $\tilde{\beta} = Cy$.

   $$\tilde{\beta} = Cy = C(X\beta + u) = CX\beta + Cu,$$

   where $C$ is a $k \times n$ matrix.

   Taking the expectation of $\tilde{\beta}$, we obtain:

   $$\mathrm{E}(\tilde{\beta}) = CX\beta + C\mathrm{E}(u) = CX\beta$$

   Because we have assumed that $\tilde{\beta} = Cy$ is unbiased, $\mathrm{E}(\tilde{\beta}) = \beta$ holds.

That is, we need the condition: $CX = I_k$.

Next, we obtain the variance of $\tilde{\beta} = Cy$.

$$\tilde{\beta} = C(X\beta + u) = \beta + Cu.$$

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}) = \mathrm{E}((\tilde{\beta} - \beta)(\tilde{\beta} - \beta)') = \mathrm{E}(Cuu'C') = \sigma^2 CC'$$

Defining $C = D + (X'X)^{-1}X'$, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\mathrm{V}(\tilde{\beta}) = \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')'.$$

Moreover, because $\hat{\beta}$ is unbiased, we have the following:

$$CX = I_k = (D + (X'X)^{-1}X')X = DX + I_k.$$

Therefore, we have the following condition:

$$DX = 0.$$

Accordingly, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\begin{aligned}\mathrm{V}(\tilde{\beta}) &= \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')' \\ &= \sigma^2 (X'X)^{-1} + \sigma^2 DD' = \mathrm{V}(\hat{\beta}) + \sigma^2 DD'\end{aligned}$$

Thus, $\mathrm{V}(\tilde{\beta}) - \mathrm{V}(\hat{\beta})$ is a positive definite matrix.

$\Longrightarrow \mathrm{V}(\tilde{\beta}_i) - \mathrm{V}(\hat{\beta}_i) > 0$

$\Longrightarrow \hat{\beta}$ is a minimum variance (i.e., best) linear unbiased estimator of $\beta$.

Note as follows:

$\Longrightarrow$ $A$ is positive definite when $d'Ad > 0$ except $d = 0$.

$\Longrightarrow$ The $i$th diagonal element of $A$, i.e., $a_{ii}$, is positive (choose $d$ such that the $i$th element of $d$ is one and the other elements are zeros).

**[Review]** $F$ **Distribution:**

Suppose that $U \sim \chi(n)$, $V \sim \chi(m)$, and $U$ is independent of $V$.
Then, $\dfrac{U/n}{V/m} \sim F(n, m)$.
**[End of Review]**

**$F$ Distribution ($H_0 : \beta = 0$):** Final Result in this Section:

$$\frac{(\hat{\beta}-\beta)X'X(\hat{\beta}-\beta)'/k}{e'e/(n-k)} \sim F(k, n-k).$$

Consider the numerator and the denominator, separately.

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$ .

   Therefore, $\dfrac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)}{\sigma^2} \sim \chi^2(k)$.

2. **Proof:**

   Using $\hat{\beta}-\beta = (X'X)^{-1}X'u$, we obtain:

$$\begin{aligned}(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta) &= ((X'X)^{-1}X'u)'X'X(X'X)^{-1}X'u \\ &= u'X(X'X)^{-1}X'X(X'X)^{-1}X'u = u'X(X'X)^{-1}X'u\end{aligned}$$

Note that $X(X'X)^{-1}X'$ is symmetric and idempotent, i.e., $A'A = A$.

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(X(X'X)^{-1}X')\Big)$$

The degree of freedom is given by:

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

Therefore, we obtain:

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

3. (*) Formula:

   Suppose that $X \sim N(0, I_k)$.

   If $A$ is symmetric and idempotent, i.e., $A'A = A$, then $X'AX \sim \chi^2(\mathrm{tr}(A))$.

   Here, $X = \dfrac{1}{\sigma}u \sim N(0, I_n)$ from $u \sim N(0, \sigma^2 I_n)$, and $A = X(X'X)^{-1}X'$.

4. **Sum of Residuals:** $e$ is rewritten as:

$$e = (I_n - X(X'X)^{-1}X')u.$$

Therefore, the sum of residuals is given by:

$$e'e = u'(I_n - X(X'X)^{-1}X')u.$$

Note that $I_n - X(X'X)^{-1}X'$ is symmetric and idempotent.

We obtain the following result:

$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(I_n - X(X'X)^{-1}X')\Big),$$

where the trace is:

$$\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k.$$

Therefore, we have the following result:

$$\frac{e'e}{\sigma^2} = \frac{(n-k)s^2}{\sigma^2} \sim \chi^2(n-k),$$

where

$$s^2 = \frac{1}{n-k}e'e.$$

5. We show that $\hat{\beta}$ is independent of $e$.

**Proof:**

Because $u \sim N(0, \sigma^2 I_n)$, we show that $\mathrm{Cov}(e, \hat{\beta}) = 0$.

$$\begin{aligned}
\mathrm{Cov}(e, \hat{\beta}) &= \mathrm{E}(e(\hat{\beta} - \beta)') = \mathrm{E}\Big((I_n - X(X'X)^{-1}X')u((X'X)^{-1}X'u)'\Big) \\
&= \mathrm{E}\Big((I_n - X(X'X)^{-1}X')uu'X(X'X)^{-1}\Big) = (I_n - X(X'X)^{-1}X')\mathrm{E}(uu')X(X'X)^{-1} \\
&= (I_n - X(X'X)^{-1}X')(\sigma^2 I_n)X(X'X)^{-1} = \sigma^2(I_n - X(X'X)^{-1}X')X(X'X)^{-1} \\
&= \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}X'X(X'X)^{-1}) = \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}) = 0.
\end{aligned}$$

$\hat{\beta}$ is independent of $e$, because of normality assumption on $u$

**[Review]**

• Suppose that $X$ is independent of $Y$. Then, $\mathrm{Cov}(X, Y) = 0$. However, $\mathrm{Cov}(X, Y) = 0$ does not mean in general that $X$ is independent of $Y$.

• In the case where $X$ and $Y$ are normal, $\mathrm{Cov}(X, Y) = 0$ indicates that $X$ is independent of $Y$.

**[End of Review]**

**[Review] Formulas — $F$ Distribution:**

- $\dfrac{U/n}{V/m} \sim F(n, m)$ when $U$ $sim\chi^2(n)$, $V \sim \chi^2(m)$, and $U$ is independent of $V$.

- When $X \sim N(0, I_n)$, $A$ and $B$ are $n \times n$ symmetric idempotent matrices, $\text{Rank}(A) = \text{tr}(A) = G$, $\text{Rank}(B) = \text{tr}(B) = K$ and $AB = 0$, then $\dfrac{X'AX/G}{X'BX/K} \sim F(G, K)$.

Note that the covariance of $AX$ and $BX$ is zero, which implies that $AX$ is independent of $BX$ under normality of $X$.

**[End of Review]**

6. Therefore, we obtain the following distribution:

$$\frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}=\frac{u'X(X'X)^{-1}X'u}{\sigma^2}\sim\chi^2(k),$$

$$\frac{e'e}{\sigma^2}=\frac{u'(I_n-X(X'X)^{-1}X')u}{\sigma^2}\sim\chi^2(n-k)$$

$\hat\beta$ is independent of $e$, because $X(X'X)^{-1}X'(I_n-X(X'X)^{-1}X')=0$.

Accordingly, we can derive:

$$\frac{\dfrac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)}=\frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)/k}{s^2}\sim F(k,n-k)$$

Under the null hypothesis $H_0:\ \beta=0$, $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2}\sim F(k,n-k)$.

Given data, $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2}$ is compared with $F(k,n-k)$.

If $\dfrac{\hat\beta'X'X\hat\beta/k}{s^2}$ is in tha tail of the $F$ distribution, the null hypothesis is rejected.

**Coefficient of Determination (決定係数), $R^2$:**

1. Definition of the Coefficient of Determination, $R^2$: $\quad R^2 = 1 - \dfrac{\sum_{i=1}^n e_i^2}{\sum_{i=1}^n (y_i - \overline{y})^2}$

2. Numerator: $\displaystyle\sum_{i=1}^n e_i^2 = e'e$

3. Denominator: $\displaystyle\sum_{i=1}^n (y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')'(I_n - \frac{1}{n}ii')y = y'(I_n - \frac{1}{n}ii')y$

   (*) Remark

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{pmatrix} = y - \frac{1}{n}ii'y = (I_n - \frac{1}{n}ii')y,$$

   where $i = (1, 1, \cdots, 1)'$.

4. In a matrix form, we can rewrite as: $R^2 = 1 - \dfrac{e'e}{y'(I_n - \frac{1}{n}ii')y}$

**$F$ Distribution and Coefficient of Determination:**

$\Longrightarrow$ This will be discussed later.

**Testing Linear Restrictions ($F$ Distribution):**

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$ .

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R : \ G \times k, \qquad \text{rank}(R) = G \leq k.$

   $R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R').$

   Therefore, $\quad \dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(G).$

   Note that $R\beta = r$ .

   (a) When $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$, the mean of $R\hat{\beta}$ is:

   $$\mathrm{E}(R\hat{\beta}) = R\mathrm{E}(\hat{\beta}) = R\beta.$$

   (b) When $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$, the variance of $R\hat{\beta}$ is:

$$V(R\hat{\beta}) = E((R\hat{\beta} - R\beta)(R\hat{\beta} - R\beta)') = E(R(\hat{\beta} - \beta)(\hat{\beta} - \beta)'R')$$

$$= RE((\hat{\beta} - \beta)(\hat{\beta} - \beta)')R' = RV(\hat{\beta})R' = \sigma^2 R(X'X)^{-1}R'.$$

2. We know that $\dfrac{(n-k)s^2}{\sigma^2} = \dfrac{e'e}{\sigma^2} = \dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{\sigma^2} \sim \chi^2(n-k)$.

3. Under normality assumption on $u$, $\hat{\beta}$ is independent of $e$.

4. Therefore, we have the following distribution:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(G, n-k)$$

5. **Some Examples:**

(a) $t$ Test:

The case of $G = 1$, $r = 0$ and $R = (0, \cdots, 1, \cdots, 0)$ (the $i$th element of $R$ is one and the other elements are zero):

The test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{(R\hat\beta - r)'(R(X'X)^{-1}R')^{-1}(R\hat\beta - r)/G}{s^2} = \frac{\hat\beta_i^2}{s^2 a_{ii}} \sim F(1, n-k),$$

where $s^2 = e'e/(n-k)$, $R\hat\beta = \hat\beta_i$ and

$$a_{ii} = R(X'X)^{-1}R' = \text{ the } i \text{ row and } i\text{th column of } (X'X)^{-1}.$$

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{\hat\beta_i}{s\sqrt{a_{ii}}} \sim t(n-k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m+1, m+2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the $F$ test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \text{rank}(R) = k$ and $\beta$ is a $2k \times 1$ vector.

The distribution is $F(k, n - 2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \text{rank}(R) = 1$

The distribution of the test statistic is $F(1, n-k)$.

(d) Testing seasonality:

In the case of **quarterly data** (四半期データ), the regression model is:

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j = 1$ in the $j$th quarter and 0 otherwise, i.e., $D_j$, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\Longrightarrow H_0: \ \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \text{rank}(R) = 3$, and $\beta$ is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n-k)$.

(e) Cobb-Douglas Production Function:

Let $Q_i$, $K_i$ and $L_i$ be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We test a linear homogeneous (一次同次) production function.

The null and alternative hypotheses are:

$$H_0: \ \beta_2 + \beta_3 = 1,$$

$$H_1: \ \beta_2 + \beta_3 \neq 1.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods $m$ and $m+1$.

In the case where both the constant term and the slope are changed, the regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m+1, m+2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m+1$.

The null and alternative hypotheses are as follows:

$H_0: \ \gamma = \delta = 0,$

$H_1: \ \gamma \neq 0$, or, $\delta \neq 0$.

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither $x_i$ nor $z_i$ depends on $y_i$.

In this case, the null and alternative hypotheses are as follows:

$$H_0 : \beta = \gamma = 0,$$

$$H_1 : \beta \neq 0, \text{ or, } \gamma \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

**Coefficient of Determination $R^2$ and $F$ distribution:**

- The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = \begin{pmatrix} 1 & x_{2i} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$$x_i : \ 1 \times k, \qquad x_{2i} : \ 1 \times (k-1), \qquad \beta : \ k \times 1, \qquad \beta_2 : \ (k-1) \times 1$$

Define:

$$X_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Then,

$$y = X\beta + u = \begin{pmatrix} i & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u = i\beta_1 + X_2\beta_2 + u,$$

where the first column of $X$ corresponds to a constant term, i.e.,

$$X = (\, i \quad X_2 \,), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- Consider testing $H_0: \ \beta_2 = 0$.

The $F$ distribution is set as follows:

$$R = (\, 0 \quad I_{k-1} \,), \qquad r = 0$$

where $R$ is a $(k-1) \times k$ matrix and $r$ is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

We are going to show:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \frac{1}{n} ii'$.

Note that $M$ is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$\begin{aligned} R(X'X)^{-1}R' &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \left( \begin{pmatrix} i' \\ X_2' \end{pmatrix} \begin{pmatrix} i & X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \end{aligned}$$

**[Review]** The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}$ and $A_{22}$ are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.

**[End of Review]**

Go back to the $F$ distribution.

$$
\begin{aligned}
\begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} &= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix}
\end{aligned}
$$

Therefore, we obtain:

$$
\begin{aligned}
&(\,0 \quad I_{k-1}\,) \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\
&= (\,0 \quad I_{k-1}\,) \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}.
\end{aligned}
$$

Thus, under $H_0: \ \beta_2 = 0$, we obtain the following result:

$$
\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} \ \sim \ F(k-1, n-k).
$$

● Coefficient of Determination $R^2$:

Define $e$ as $e = y - X\hat{\beta}$. The coefficient of determinant, $R^2$, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \dfrac{1}{n}ii'$, $I_n$ is a $n \times n$ identity matrix and $i$ is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = (\, i \quad X_2 \,)$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

because $i'e = 0$, and

$$MX = M(\, i \quad X_2 \,) = (\, Mi \quad MX_2 \,) = (\, 0 \quad MX_2 \,),$$

because $Mi = 0$.

$$MX\hat{\beta} = \begin{pmatrix} 0 & MX_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = MX_2\hat{\beta}_2.$$

Thus,

$$My = MX\hat{\beta} + Me \qquad \Longrightarrow \qquad My = MX_2\hat{\beta}_2 + e.$$

$y'My$ is given by: $y'My = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2 + e'e$, because $X_2'e = 0$ and $Me = e$.

The coefficient of determinant, $R^2$, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \qquad \Longrightarrow \qquad e'e = (1 - R^2) y'My,$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2}{y'My} \qquad \Longrightarrow \qquad \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2 = R^2 y'My.$$

Therefore,

$$\frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2 / (k-1)}{e'e/(n-k)} = \frac{R^2 y'My/(k-1)}{(1-R^2) y'My/(n-k)} = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k).$$

Thus, using $R^2$, the null hypothesis $H_0: \beta_2 = 0$ is easily tested.

# 5 Restricted OLS (制約付き最小二乗法)

1. Let $\tilde{\beta}$ be the restricted estimator.

   Consider the linear restriction: $R\beta = r$.

2. Minimize $(y - X\tilde{\beta})'(y - X\tilde{\beta})$ subject to $R\tilde{\beta} = r$.

   Let $L$ be the Lagrangian for the minimization problem.

$$L = (y - X\tilde{\beta})'(y - X\tilde{\beta}) - 2\tilde{\lambda}'(R\tilde{\beta} - r)$$

   Because $\tilde{\beta}$ and $\tilde{\lambda}$ minimize the Lagrangian $L$,

$$\begin{aligned}
\frac{\partial L}{\partial \tilde{\beta}} &= -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0 \\
\frac{\partial L}{\partial \tilde{\lambda}} &= -2(R\tilde{\beta} - r) = 0.
\end{aligned}$$

(*) Remember that $\dfrac{\partial a'x}{\partial x} = a$ and $\dfrac{\partial x'Ax}{\partial x} = (A + A')x$.

From $\dfrac{\partial L}{\partial \tilde{\beta}} = 0$, we obtain:

$$\tilde{\beta} = (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}.$$

Multiplying $R$ from the left, we have:

$$R\tilde{\beta} = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Because $R\tilde{\beta} = r$ has to be satisfied, we have the following expression:

$$r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Therefore, solving the above equation with respect to $\tilde{\lambda}$, we obtain:

$$\tilde{\lambda} = \left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta})$$

Substituting $\tilde{\lambda}$ into $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}$, the restricted OLSE is given by:

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat{\beta}).$$

(a) The expectation of $\tilde{\beta}$ is:

$$\begin{aligned} \mathrm{E}(\tilde{\beta}) &= \mathrm{E}(\hat{\beta}) + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\mathrm{E}(\hat{\beta})) \\ &= \beta + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\beta) \\ &= \beta, \end{aligned}$$

because of $R\beta = r$.

Thus, it is shown that $\tilde{\beta}$ is unbiased.

(b) The variance of $\tilde{\beta}$ is as follows.

First, rewrite as follows:

$$
\begin{aligned}
(\tilde{\beta}-\beta) &= (\hat{\beta}-\beta) + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\beta - R\hat{\beta}) \\
&= (\hat{\beta}-\beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - R\beta) \\
&= (\hat{\beta}-\beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(\hat{\beta} - \beta) \\
&= \left(I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\right)(\hat{\beta} - \beta) \\
&= W(\hat{\beta}-\beta),
\end{aligned}
$$

where $W \equiv I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R$.

Then, we obtain the following variance:

$$
\begin{aligned}
\mathrm{V}(\tilde{\beta}) &\equiv \mathrm{E}((\tilde{\beta}-\beta)(\tilde{\beta}-\beta)') = \mathrm{E}(W(\hat{\beta}-\beta)(\hat{\beta}-\beta)'W') \\
&= W\mathrm{E}((\hat{\beta}-\beta)(\hat{\beta}-\beta)')W' = W\mathrm{V}(\hat{\beta})W' = \sigma^2 W(X'X)^{-1}W'
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^2\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)(X'X)^{-1} \\
&\qquad\qquad\qquad\qquad \times\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)' \\
&= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1} \\
&= \mathrm{V}(\hat{\beta}) - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}
\end{aligned}
$$

Thus, $\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta})$ is positive definite.

3. Another solution:

Again, write the first-order condition for minimization:

$$\frac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0,$$
$$\frac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0,$$

which can be written as:

$$X'X\tilde{\beta} - R'\tilde{\lambda} = X'y,$$
$$R\tilde{\beta} = r.$$

Using the matrix form:

$$\begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix} \begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'y \\ r \end{pmatrix}.$$

The solutions of $\tilde{\beta}$ and $-\tilde{\lambda}$ are given by:

$$\begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1} \begin{pmatrix} X'y \\ r \end{pmatrix}.$$

(*) Formula to the inverse matrix:

$$\begin{pmatrix} A & B \\ B' & D \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ F' & G \end{pmatrix},$$

where $E$, $F$ and $G$ are given by:

$$E = (A - BD^{-1}B')^{-1} = A^{-1} + A^{-1}B(D - B'A^{-1}B)^{-1}B'A^{-1}$$

$$F = -(A - BD^{-1}B')^{-1}BD^{-1} = -A^{-1}B(D - B'A^{-1}B)^{-1}$$

$$G = (D - B'A^{-1}B)^{-1} = D^{-1} + D^{-1}B'(A - BD^{-1}B')^{-1}BD^{-1}$$

In this case, $E$ and $F$ correspond to:

$$
\begin{aligned}
E &= (X'X)^{-1} - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R(X'X)^{-1} \\
F &= (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}.
\end{aligned}
$$

Therefore, $\tilde{\beta}$ is derived as follows:

$$
\begin{aligned}
\tilde{\beta} &= EX'y + Fr \\
&= \hat{\beta} + (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat{\beta}).
\end{aligned}
$$

The variance is:

$$
\mathrm{V}\begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \sigma^2 \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1}.
$$

Therefore, $\mathrm{V}(\tilde{\beta})$ is:

$$
\mathrm{V}(\tilde{\beta}) = \sigma^2 E = \sigma^2\Big((X'X)^{-1} - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R(X'X)^{-1}\Big)
$$

Under the restriction: $R\beta = r$,

$$\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta}) = \sigma^2 (X'X)^{-1} R' \Big( R(X'X)^{-1} R' \Big)^{-1} R(X'X)^{-1}$$

is positive definite.

# 6 *F* Distribution (Restricted and Unrestricted OLSs)

1. As mentioned above, under the null hypothesis $H_0 : R\beta = r$,

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(G, n-k),$$

where $G = \mathrm{Rank}(R)$.

Using $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R' \Big( R(X'X)^{-1}R' \Big)^{-1} (r - R\hat{\beta})$, the numerator is rewritten as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = (\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta}).$$

Moreover, the numerator is represented as follows:

$$
\begin{aligned}
(y - X\tilde{\beta})'(y - X\tilde{\beta}) =& (y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}))'(y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta})) \\
=& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\
& -(y - X\hat{\beta})'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'(y - X\hat{\beta}) \\
=& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}).
\end{aligned}
$$

$X'(y - X\hat{\beta}) = X'e = 0$ is utilized.

Summarizing, we have following representation:

$$\begin{aligned}(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r) &= (\tilde{\beta}-\hat{\beta})'X'X(\tilde{\beta}-\hat{\beta}) \\ &= (y-X\tilde{\beta})'(y-X\tilde{\beta})-(y-X\hat{\beta})'(y-X\hat{\beta}) \\ &= \tilde{u}'\tilde{u}-e'e,\end{aligned}$$

where $e$ and $\tilde{u}$ are the restricted residual and the unrestricted residual, i.e., $e = y - X\hat{\beta}$ and $\tilde{u} = y - X\tilde{\beta}$.

Therefore, we obtain the following result:

$$\frac{(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)/G}{(y-X\hat{\beta})'(y-X\hat{\beta})/(n-k)} = \frac{(\tilde{u}'\tilde{u}-e'e)/G}{e'e/(n-k)} \sim F(G, n-k).$$

# 7 Example: $F$ Distribution (Restricted OLS and Unrestricted OLS)

Date file $\Longrightarrow$ cons99.txt (Next slide)

Each column denotes year, nominal household expenditures (家計消費，10 billion yen), household disposable income (家計可処分所得，10 billion yen) and household expenditure deflator (家計消費デフレータ，1990=100) from the left.

| | | | |
|---|---|---|---|
| 1955 | 5430.1 | 6135.0 | 18.1 |
| 1956 | 5974.2 | 6828.4 | 18.3 |
| 1957 | 6686.3 | 7619.5 | 19.0 |
| 1958 | 7169.7 | 8153.3 | 19.1 |
| 1959 | 8019.3 | 9274.3 | 19.7 |
| 1960 | 9234.9 | 10776.5 | 20.5 |
| 1961 | 10836.2 | 12869.4 | 21.8 |
| 1962 | 12430.8 | 14701.4 | 23.2 |
| 1963 | 14506.6 | 17042.7 | 24.9 |
| 1964 | 16674.9 | 19709.9 | 26.0 |
| 1965 | 18820.5 | 22337.4 | 27.8 |
| 1966 | 21680.6 | 25514.5 | 29.0 |
| 1967 | 24914.0 | 29012.6 | 30.1 |
| 1968 | 28452.7 | 34233.6 | 31.6 |
| 1969 | 32705.2 | 39486.3 | 32.9 |
| 1970 | 37784.1 | 45913.2 | 35.2 |
| 1971 | 42571.6 | 51944.3 | 37.5 |
| 1972 | 49124.1 | 60245.4 | 39.7 |
| 1973 | 59366.1 | 74924.8 | 44.1 |
| 1974 | 71782.1 | 93833.2 | 53.3 |
| 1975 | 83591.1 | 108712.8 | 59.4 |
| 1976 | 94443.7 | 123540.9 | 65.2 |
| 1977 | 105397.8 | 135318.4 | 70.1 |
| 1978 | 115960.3 | 147244.2 | 73.5 |
| 1979 | 127600.9 | 157071.1 | 76.0 |
| 1980 | 138585.0 | 169931.5 | 81.6 |
| 1981 | 147103.4 | 181349.2 | 85.4 |
| 1982 | 157994.0 | 190611.5 | 87.7 |
| 1983 | 166631.6 | 199587.8 | 89.5 |
| 1984 | 175383.4 | 209451.9 | 91.8 |
| 1985 | 185335.1 | 220655.6 | 93.9 |
| 1986 | 193069.6 | 229938.8 | 94.8 |
| 1987 | 202072.8 | 235924.0 | 95.3 |
| 1988 | 212939.9 | 247159.7 | 95.8 |
| 1989 | 227122.2 | 263940.5 | 97.7 |
| 1990 | 243035.7 | 280133.0 | 100.0 |
| 1991 | 255531.8 | 297512.9 | 102.5 |
| 1992 | 265701.6 | 309256.6 | 104.5 |
| 1993 | 272075.3 | 317021.6 | 105.9 |
| 1994 | 279538.7 | 325655.7 | 106.7 |
| 1995 | 283245.4 | 331967.5 | 106.2 |
| 1996 | 291458.5 | 340619.1 | 106.0 |
| 1997 | 298475.2 | 345522.7 | 107.3 |

Estimate using TSP 5.0.

```
LINE   ***********************************************
|      1   freq a;
|      2   smpl 1955 1997;
|      3   read(file='cons99.txt') year cons yd price;
|      4   rcons=cons/(price/100);
|      5   ryd=yd/(price/100);
|      6   d1=0.0;
|      7   smpl 1974 1997;
|      8   d1=1.0;
|      9   smpl 1956 1997;
|     10   d1ryd=d1*ryd;
|     11   olsq rcons c ryd;
|     12   olsq rcons c d1 ryd d1ryd;
|     13   end;
*****************************************************
```

```
                 Equation   1
                 ============

   Method of estimation = Ordinary Least Squares


 Dependent variable: RCONS
 Current sample:  1956 to 1997
 Number of observations:  42

  Mean of dependent variable = 149038.
 Std. dev. of dependent var. = 78147.9
    Sum of squared residuals = .127951E+10
       Variance of residuals = .319878E+08
    Std. error of regression = 5655.77
                   R-squared = .994890
          Adjusted R-squared = .994762
     Durbin-Watson statistic = .116873
   F-statistic (zero slopes) = 7787.70
  Schwarz Bayes. Info. Crit. = 17.4101
  Log of likelihood function = -421.469

           Estimated    Standard
```

| Variable | Coefficient | Error | t-statistic |
|---|---|---|---|
| C | -3317.80 | 1934.49 | -1.71508 |
| RYD | .854577 | .968382E-02 | 88.2480 |

```
                Equation   2
                ============

  Method of estimation = Ordinary Least Squares


Dependent variable: RCONS
Current sample:  1956 to 1997
Number of observations:  42

 Mean of dependent variable = 149038.
Std. dev. of dependent var. = 78147.9
   Sum of squared residuals = .244501E+09
      Variance of residuals = .643423E+07
   Std. error of regression = 2536.58
                  R-squared = .999024
         Adjusted R-squared = .998946
    Durbin-Watson statistic = .420979
  F-statistic (zero slopes) = 12959.1
 Schwarz Bayes. Info. Crit. = 15.9330
 Log of likelihood function = -386.714

           Estimated    Standard
```

| Variable | Coefficient | Error | t-statistic |
|---|---|---|---|
| C | 4204.11 | 1440.45 | 2.91861 |
| D1 | -39915.3 | 3154.24 | -12.6545 |
| RYD | .786609 | .015024 | 52.3561 |
| D1RYD | .194495 | .018731 | 10.3839 |

1. `Equation 1`

Significance test:

`Equation 1` is:

$$\texttt{RCONS} = \beta_1 + \beta_2 \texttt{RYD}$$

$H_0: \ \beta_2 = 0$

(No.1) $t$ Test $\Longrightarrow$ Compare 88.2480 and $t(42-2)$.

(No.2) $F$ Test $\Longrightarrow$ Compare $\dfrac{R^2/G}{(1-R^2)/(n-k)} = \dfrac{.994890/1}{(1-.994890)/(42-2)} =$ 7787.8 and $F(1,40)$. Note that $\sqrt{7787.8} = 88.2485$.

1% point of $F(1,40) = 7.31$

$H_0: \ \beta_2 = 0$ is rejected.

2. `Equation 2`:

$$\mathtt{RCONS} = \beta_1 + \beta_2\mathtt{D1} + \beta_3\mathtt{RYD} + \beta_4\mathtt{RYD}\times\mathtt{D1}$$

$H_0:\ \beta_2 = \beta_3 = \beta_4 = 0$

$F$ Test $\Longrightarrow$ Compare $\dfrac{R^2/G}{(1-R^2)/(n-k)} = \dfrac{.999024/3}{(1-.999024)/(42-4)} = 12965.5$ and $F(3,38)$.

1% point of $F(3,38) = 4.34$

$H_0:\ \beta_2 = \beta_3 = \beta_4 = 0$ is rejected.

3. `Equation 1` vs. `Equation 2`

Test the structural change between 1973 and 1974.

`Equation 2` is:

$$\texttt{RCONS} = \beta_1 + \beta_2\texttt{D1} + \beta_3\texttt{RYD} + \beta_4\texttt{RYD} \times \texttt{D1}$$

$H_0:\ \beta_2 = \beta_4 = 0$

Restricted OLS $\Longrightarrow$ `Equation 1`

Unrestricted OLS $\Longrightarrow$ `Equation 2`

$$\frac{(\tilde{u}'\tilde{u} - e'e)/G}{e'e/(n-k)} = \frac{(.127951\text{E}+10 - .244501\text{E}+09)/2}{.244501\text{E}+09/(42-4)} = 80.43$$

which should be compared with $F(2, 38)$.

1% point of $F(2, 38) = 5.211 < 80.43$

$H_0:\ \beta_2 = \beta_4 = 0$ is rejected.

$\Longrightarrow$ The structure was changed in 1974.

# 8 Generalized Least Squares Method (GLS, 一般化最小自乗法)

1. Regression model: $y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega)$

2. **Heteroscedasticity** (不等分散，不均一分散)

$$\sigma^2\Omega = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n^2 \end{pmatrix}$$

**First-Order Autocorrelation (一階の自己相関，系列相関)**

In the case of time series data, the subscript is conventionally given by $t$, not $i$ .

$$u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

$$\sigma^2 \Omega = \frac{\sigma_\epsilon^2}{1-\rho^2} \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$

$$\mathrm{V}(u_t) = \sigma^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$$

3. The Generalized Least Squares (GLS，一般化最小二乗法) estimator of $\beta$,

denoted by $b$, solves the following minimization problem:

$$\min_{b} \ (y - Xb)'\Omega^{-1}(y - Xb)$$

The GLSE of $\beta$ is:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

4. In general, when $\Omega$ is symmetric, $\Omega$ is decomposed as follows.

$$\Omega = A'\Lambda A$$

$\Lambda$ is a diagonal matrix, where the diagonal elements of $\Lambda$ are given by the eigen values.

$A$ is a matrix consisting of eigen vectors.

When $\Omega$ is a positive definite matrix, all the diagonal elements of $\Lambda$ are positive.

5. There exists $P$ such that $\Omega = PP'$ (i.e., take $P = A'\Lambda^{1/2}$). $\Longrightarrow$ $P^{-1}\Omega P'^{-1} = I_n$

Multiply $P^{-1}$ on both sides of $y = X\beta + u$.

We have:

$$y^\star = X^\star\beta + u^\star,$$

where $\quad y^\star = P^{-1}y, \quad X^\star = P^{-1}X, \quad$ and $\quad u^\star = P^{-1}u.$

The variance of $u^\star$ is:

$$\mathrm{V}(u^\star) = \mathrm{V}(P^{-1}u) = P^{-1}\mathrm{V}(u)P'^{-1} = \sigma^2 P^{-1}\Omega P'^{-1} = \sigma^2 I_n.$$

because $\Omega = PP'$, i.e., $P^{-1}\Omega P'^{-1} = I_n$.

Accordingly, the regression model is rewritten as:

$$y^\star = X^\star\beta + u^\star, \qquad u^\star \sim (0, \sigma^2 I_n)$$

Apply OLS to the above model.

Let $b$ be as estimator of $\beta$ from the above model.

That is, the minimization problem is given by:

$$\min_{b} \ (y^\star - X^\star b)'(y^\star - X^\star b),$$

which is equivalent to:

$$\min_{b} \ (y - Xb)'\Omega^{-1}(y - Xb).$$

Solving the minimization problem above, we have the following estimator:

$$\begin{aligned} b &= (X^{\star\prime} X^\star)^{-1} X^{\star\prime} y^\star \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y, \end{aligned}$$

which is called GLS (Generalized Least Squares) estimator.

$b$ is rewritten as follows:

$$b = \beta + (X^{\star\prime}X^{\star})^{-1}X^{\star\prime}u^{\star} = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u$$

The mean and variance of $b$ are given by:

$$\mathrm{E}(b) = \beta,$$

$$\mathrm{V}(b) = \sigma^2(X^{\star\prime}X^{\star})^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}.$$

6. Suppose that the regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

In this case, when we use OLS, what happens?

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$$

$$\mathrm{V}(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}$$

Compare GLS and OLS.

(a) Expectation:

$$\mathrm{E}(\hat{\beta}) = \beta, \quad \text{and} \quad \mathrm{E}(b) = \beta$$

Thus, both $\hat{\beta}$ and $b$ are unbiased estimator.

(b) Variance:

$$\mathrm{V}(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}$$

$$\mathrm{V}(b) = \sigma^2 (X'\Omega^{-1} X)^{-1}$$

Which is more efficient, OLS or GLS?.

$$
\begin{aligned}
\mathrm{V}(\hat{\beta}) - \mathrm{V}(b) &= \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1} - \sigma^2 (X'\Omega^{-1}X)^{-1} \\
&= \sigma^2 \Big( (X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1} \Big) \Omega \\
&\qquad\qquad \times \Big( (X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1} \Big)' \\
&= \sigma^2 A\Omega A'
\end{aligned}
$$

$\Omega$ is the variance-covariance matrix of $u$, which is a positive definite matrix.

Therefore, except for $\Omega = I_n$, $A\Omega A'$ is also a positive definite matrix.

This implies that $\mathrm{V}(\hat{\beta}_i) - \mathrm{V}(b_i) > 0$ for the $i$th element of $\beta$.

Accordingly, $b$ is more efficient than $\hat{\beta}$.

7. If $u \sim N(0, \sigma^2\Omega)$, then $b \sim N(\beta, \sigma^2(X'\Omega^{-1}X)^{-1})$.

Consider testing the hypothesis $H_0 : R\beta = r$.

$R: \; G \times k, \quad \text{rank}(R) = G \leq k.$

$Rb \sim N(R\beta, \sigma^2 R(X'\Omega^{-1}X)^{-1}R').$

Therefore, the following quadratic form is distributed as:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)}{\sigma^2} \sim \chi^2(G)$$

8. Because $(y^\star - X^\star b)'(y^\star - X^\star b)/\sigma^2 \sim \chi^2(n - k)$, we obtain:

$$\frac{(y - Xb)'\Omega^{-1}(y - Xb)}{\sigma^2} \sim \chi^2(n - k)$$

9. Furthermore, from the fact that $b$ is independent of $y - Xb$, the following $F$ distribution can be derived:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)/G}{(y - Xb)'\Omega^{-1}(y - Xb)/(n - k)} \sim F(G, n - k)$$

10. Let $b$ be the unrestricted GLSE and $\tilde{b}$ be the restricted GLSE.

    Their residuals are given by $e$ and $\tilde{u}$, respectively.

$$e = y - Xb, \qquad \tilde{u} = y - X\tilde{b}$$

    Then, the $F$ test statistic is written as follows:

$$\frac{(\tilde{u}'\Omega^{-1}\tilde{u} - e'\Omega^{-1}e)/G}{e'\Omega^{-1}e/(n - k)} \sim F(G, n - k)$$

## 8.1 Example: Mixed Estimation (Theil and Goldberger Model)

A generalization of the restricted OLS $\Longrightarrow$ Stochastic linear restriction:

$$r = R\beta + v, \qquad \mathrm{E}(v) = 0 \text{ and } \mathrm{V}(v) = \sigma^2\Psi$$

$$y = X\beta + u, \qquad \mathrm{E}(u) = 0 \text{ and } \mathrm{V}(u) = \sigma^2 I_n$$

Using a matrix form,

$$\begin{pmatrix} y \\ r \end{pmatrix} = \begin{pmatrix} X \\ R \end{pmatrix}\beta + \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathrm{E}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathrm{V}\begin{pmatrix} u \\ v \end{pmatrix} = \sigma^2\begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}$$

For estimation, we do not need normality assumption.

Applying GLS, we obtain:

$$\begin{aligned} b &= \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right) \\ &= \left( X'X + R'\Psi^{-1}R \right)^{-1} \left( X'y + R'\Psi^{-1}r \right). \end{aligned}$$

Mean and Variance of $b$: $b$ is rewritten as follows:

$$b = \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right)$$

$$= \beta + \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

Therefore, the mean and variance are given by:

$$\mathrm{E}(b) = \beta \qquad \Longrightarrow \qquad b \text{ is unbiased.}$$

$$\mathrm{V}(b) = \sigma^2 \left( \begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1}$$

$$= \sigma^2 \left( X'X + R'\Psi^{-1}R \right)^{-1}$$

# 9 Maximum Likelihood Estimation (MLE, 最尤法)

$\longrightarrow$ **Review**

1. The distribution function of $\{X_i\}_{i=1}^n$ is $f(x;\theta)$, where $x = (x_1, x_2, \cdots, x_n)$ and $\theta = (\mu, \Sigma)$.

   Note that $X$ is a vector of random variables and $x$ is a vector of their realizations (i.e., observed data).

   Likelihood function $L(\cdot)$ is defined as $L(\theta; x) = f(x;\theta)$.

   Note that $f(x;\theta) = \prod_{i=1}^n f(x_i;\theta)$ when $X_1$, $X_2$, $\cdots$, $X_n$ are mutually independently and identically distributed.

The maximum likelihood estimator (MLE) of $\theta$ is $\theta$ such that:

$$\max_{\theta} \; L(\theta; X). \qquad \Longleftrightarrow \qquad \max_{\theta} \; \log L(\theta; X).$$

MLE satisfies the following two conditions:

(a) $\dfrac{\partial \log L(\theta; X)}{\partial \theta} = 0.$

(b) $\dfrac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}$ is a negative definite matrix.

2. **Fisher's information matrix (フィッシャーの情報行列)** is defined as:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\Big),$$

where we have the following equality:

$$-\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\Big) = \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta} \frac{\partial \log L(\theta; X)}{\partial \theta'}\Big) = \mathrm{V}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big)$$

**Proof of the above equality:**

$$\int L(\theta; x)\mathrm{d}x = 1$$

Take a derivative with respect to $\theta$.

$$\int \frac{\partial L(\theta; x)}{\partial \theta}\mathrm{d}x = 0$$

(We assume that (i) the domain of $x$ does not depend on $\theta$ and (ii) the derivative $\dfrac{\partial L(\theta; x)}{\partial \theta}$ exists.)

Rewriting the above equation, we obtain:

$$\int \frac{\partial \log L(\theta; x)}{\partial \theta} L(\theta; x)\mathrm{d}x = 0,$$

i.e.,

$$\mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right) = 0.$$

Again, differentiating the above with respect to $\theta$, we obtain:

$$\int \frac{\partial^2 \log L(\theta; x)}{\partial\theta\partial\theta'} L(\theta; x)\mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial\theta} \frac{\partial L(\theta; x)}{\partial'\theta}\mathrm{d}x$$

$$= \int \frac{\partial^2 \log L(\theta; x)}{\partial\theta\partial\theta'} L(\theta; x)\mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial\theta} \frac{\partial \log L(\theta; x)}{\partial\theta'} L(\theta; x)\mathrm{d}x$$

$$= \mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\Big) + \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial\theta} \frac{\partial \log L(\theta; X)}{\partial\theta'}\Big) = 0.$$

Therefore, we can derive the following equality:

$$-\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\right) = \mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial\theta} \frac{\partial \log L(\theta; X)}{\partial\theta'}\right) = \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right),$$

where the second equality utilizes $\mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right) = 0.$

3. **Cramer-Rao Lower Bound (**クラメール・ラオの下限**)**: $(I(\theta))^{-1}$

Suppose that an estimator of $\theta$ is given by $s(X)$.

The expectation of $s(X)$ is:

$$\mathrm{E}(s(X)) = \int s(x)L(\theta; x)\mathrm{d}x.$$

Differentiating the above with respect to $\theta$,

$$\begin{aligned}\frac{\partial \mathrm{E}(s(X))}{\partial \theta} &= \int s(x)\frac{\partial L(\theta; x)}{\partial \theta}\mathrm{d}x = \int s(x)\frac{\partial \log L(\theta; x)}{\partial \theta}L(\theta; x)\mathrm{d}x \\ &= \mathrm{Cov}\left(s(X), \frac{\partial \log L(\theta; X)}{\partial \theta}\right)\end{aligned}$$

For simplicity, let $s(X)$ and $\theta$ be scalars.

Then,

$$\begin{aligned}\left(\frac{\partial \mathrm{E}(s(X))}{\partial \theta}\right)^2 &= \left(\mathrm{Cov}\left(s(X), \frac{\partial \log L(\theta; X)}{\partial \theta}\right)\right)^2 = \rho^2 \mathrm{V}\left(s(X)\right)\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right) \\ &\leq \mathrm{V}\left(s(X)\right)\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right),\end{aligned}$$

where $\rho$ denotes the correlation coefficient between $s(X)$ and $\dfrac{\partial \log L(\theta; X)}{\partial\theta}$, i.e.,

$$\rho = \frac{\mathrm{Cov}\left(s(X), \dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}{\sqrt{\mathrm{V}\left(s(X)\right)}\sqrt{\mathrm{V}\left(\dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}}.$$

Note that $|\rho| \le 1$.

Therefore, we have the following inequality:

$$\left(\frac{\partial \mathrm{E}(s(X))}{\partial\theta}\right)^2 \le \mathrm{V}(s(X))\,\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right),$$

i.e.,

$$\mathrm{V}(s(X)) \ge \frac{\left(\dfrac{\partial \mathrm{E}(s(X))}{\partial\theta}\right)^2}{\mathrm{V}\left(\dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}$$

Especially, when $\mathrm{E}(s(X)) = \theta$,

$$\mathrm{V}(s(X)) \geq \frac{1}{-\mathrm{E}\left(\dfrac{\partial^2 \log L(\theta; X)}{\partial \theta^2}\right)} = (I(\theta))^{-1}.$$

Even in the case where $s(X)$ is a vector, the following inequality holds.

$$\mathrm{V}(s(X)) \geq (I(\theta))^{-1},$$

where $I(\theta)$ is defined as:

$$\begin{aligned}
I(\theta) &= -\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\right) \\
&= \mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta} \frac{\partial \log L(\theta; X)}{\partial \theta'}\right) = \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right).
\end{aligned}$$

The variance of any unbiased estimator of $\theta$ is larger than or equal to $(I(\theta))^{-1}$.

4. Asymptotic Normality of MLE:

Let $\tilde{\theta}$ be MLE of $\theta$.

As $n$ goes to infinity, we have the following result:

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N\left(0, \lim_{n\to\infty}\left(\frac{I(\theta)}{n}\right)^{-1}\right),$$

where it is assumed that $\lim_{n\to\infty}\left(\frac{I(\theta)}{n}\right)$ converges.

That is, when $n$ is large, $\tilde{\theta}$ is approximately distributed as follows:

$$\tilde{\theta} \sim N\left(\theta, (I(\theta))^{-1}\right).$$

Suppose that $s(X) = \tilde{\theta}$.

When $n$ is large, $\mathrm{V}(s(X))$ is approximately equal to $(I(\theta))^{-1}$.

5. **Optimization (最適化):**

MLE of $\theta$ results in the following maximization problem:

$$\max_{\theta} \quad \log L(\theta; x).$$

We often have the case where the solution of $\theta$ is not derived in closed form.

$\Longrightarrow$ Optimization procedure

$$0 = \frac{\partial \log L(\theta; x)}{\partial \theta} = \frac{\partial \log L(\theta^*; x)}{\partial \theta} + \frac{\partial^2 \log L(\theta^*; x)}{\partial \theta \partial \theta'}(\theta - \theta^*).$$

Solving the above equation with respect to $\theta$, we obtain the following:

$$\theta = \theta^* - \left(\frac{\partial^2 \log L(\theta^*; x)}{\partial \theta \partial \theta'}\right)^{-1} \frac{\partial \log L(\theta^*; x)}{\partial \theta}.$$

Replace the variables as follows:

$$\theta \longrightarrow \theta^{(i+1)}$$

$$\theta^* \longrightarrow \theta^{(i)}$$

Then, we have:

$$\theta^{(i+1)} = \theta^{(i)} - \left(\frac{\partial^2 \log L(\theta^{(i)}; x)}{\partial\theta\partial\theta'}\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial\theta}.$$

$\Longrightarrow$ **Newton-Raphson method (ニュートン・ラプソン法)**

Replacing $\dfrac{\partial^2 \log L(\theta^{(i)}; x)}{\partial\theta\partial\theta'}$ by $\mathrm{E}\left(\dfrac{\partial^2 \log L(\theta^{(i)}; x)}{\partial\theta\partial\theta'}\right)$, we obtain the following optimization algorithm:

$$\begin{aligned}
\theta^{(i+1)} &= \theta^{(i)} - \left(\mathrm{E}\left(\frac{\partial^2 \log L(\theta^{(i)}; x)}{\partial\theta\partial\theta'}\right)\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial\theta} \\
&= \theta^{(i)} + \left(I(\theta^{(i)})\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial\theta}
\end{aligned}$$

$\Longrightarrow$ **Method of Scoring (スコア法)**

## 9.1 MLE: The Case of Single Regression Model

The regression model:

$$y_i = \beta_1 + \beta_2 x_i + u_i,$$

1. $u_i \sim N(0, \sigma^2)$ is assumed.

2. The density function of $u_i$ is:

$$f(u_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} u_i^2\right).$$

   Because $u_1, u_2, \cdots, u_n$ are mutually independently distributed, the joint density function of $u_1, u_2, \cdots, u_n$ is written as:

$$\begin{aligned} f(u_1, u_2, \cdots, u_n) &= f(u_1) f(u_2) \cdots f(u_n) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} u_i^2\right) \end{aligned}$$

3. Using the transformation of variable ($u_i = y_i - \beta_1 - \beta_2 x_i$), the joint density function of $y_1, y_2, \cdots, y_n$ is given by:

$$\begin{aligned} f(y_1, y_2, \cdots, y_n) &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2\right) \\ &\equiv L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n). \end{aligned}$$

$L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$ is called the likelihood function.

$\log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$ is called the log-likelihood function.

$$\begin{aligned} &\log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n) \\ &\qquad = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_t - \beta_1 - \beta_2 x_i)^2 \end{aligned}$$

4. **Transformation of Variable (変数変換) — Review**:

Suppose that the density function of a random variable $X$ is $f_x(x)$.

Defining $X = g(Y)$, the density function of $Y$, $f_y(y)$, is given by:

$$f_y(y) = f_x(g(y)) \left| \frac{\mathrm{d}g(y)}{\mathrm{d}y} \right| .$$

In the case where $X$ and $g(Y)$ are $n \times 1$ vectors, $\left| \dfrac{\mathrm{d}g(y)}{\mathrm{d}y} \right|$ should be replaced by $\left| \dfrac{\partial g(y)}{\partial y'} \right|$, which is an absolute value of a determinant of the matrix $\dfrac{\partial g(y)}{\partial y'}$.

**Example:** When $X \sim U(0,1)$, derive the density function of $Y = -\log(X)$.

$$f_x(x) = 1$$

$X = \exp(-Y)$ is obtained.

Therefore, the density function of $Y$, $f_y(y)$, is given by:

$$f_y(y) = \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| f_x(g(y)) = |-\exp(-y)| = \exp(-y)$$

5. Given the observed data $y_1, y_2, \cdots, y_n$, the likelihood function $L(\beta_1,\ \beta_2,\ \sigma^2|y_1,\ y_2,\ \cdots,\ y_n)$, or the log-likelihood function $\log L(\beta_1,\ \beta_2,\ \sigma^2|y_1,\ y_2,\ \cdots,\ y_n)$ is maximized with respect to $(\beta_1, \beta_2, \sigma^2)$.

   Solve the following three simultaneous equations:

$$\frac{\partial \log L(\beta_1, \beta_2, \sigma^2|y_1, y_2, \cdots, y_n)}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) = 0,$$

$$\frac{\partial \log L(\beta_1, \beta_2, \sigma^2|y_1, y_2, \cdots, y_n)}{\partial \beta_2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)x_i = 0,$$

$$\frac{\partial \log L(\beta_1, \beta_2, \sigma^2|y_1, y_2, \cdots, y_n)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 = 0.$$

   The solutions of $(\beta_1,\ \beta_2,\ \sigma^2)$ are called the maximum likelihood estimates, denoted by $(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\sigma}^2)$.

The maximum likelihood estimates are:

$$\tilde{\beta}_2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2}, \qquad \tilde{\beta}_1 = \overline{y} - \tilde{\beta}_2\overline{x}, \qquad \tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{\beta}_1 - \tilde{\beta}_2 x_i)^2.$$

The MLE of $\sigma^2$ is divided by $n$, not $n - 2$.

## 9.2 MLE: The Case of Multiple Regression Model I

1. Multivariate Normal Distribution: $X:\ n\times 1$ and $X \sim N(\mu,\Sigma)$

   The density function of $X$ is:

$$f(x) = (2\pi)^{n/2}|\Sigma|^{-1/2}\exp\Big(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big).$$

2. Regression model: $y = X\beta + u,\quad u \sim N(0,\sigma^2 I_n)$

   Transformation of Variables from $u$ to $y$:

$$f_u(u) = (2\pi\sigma^2)^{-n/2}\exp\Big(-\frac{1}{2\sigma^2}u'u\Big)$$

$$\begin{aligned} f_y(y) &= f_u(y - X\beta)\left|\frac{\partial u}{\partial y'}\right| \\ &= (2\pi\sigma^2)^{-n/2}\exp\Big(-\frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)\Big) \\ &= L(\theta; y, X), \end{aligned}$$

where $\theta = (\beta, \sigma^2)$, because of $\dfrac{\partial u}{\partial y'} = I_n$.

Therefore, the log-likelihood function is:

$$\log L(\theta; y, X) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta),$$

Note that $|\Sigma|^{-1/2} = |\sigma^2 I_n|^{-1/2} = \sigma^{-n/2}$.

3. $\max\limits_{\theta} \quad \log L(\theta; y, X)$

$$\text{(FOC)} \quad \frac{\partial \log L(\theta; y, X)}{\partial \theta} = 0$$

$$\text{(SOC)} \quad \frac{\partial^2 \log L(\theta; y, X)}{\partial \theta \partial \theta'} \text{ is a negative definite matrix.}$$

We obtain MLE of $\beta$ and $\sigma^2$:

$$\tilde{\beta} = (X'X)^{-1}X'y, \qquad \tilde{\sigma}^2 = \frac{(y - X\tilde{\beta})'(y - X\tilde{\beta})}{n},$$

where $\tilde{\sigma}^2$ is divided by $n$, not $n - k$.

4. Fisher's information matrix is:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; y, X)}{\partial\theta\partial\theta'}\Big)$$

The inverse of the information matrix, $I(\theta)^{-1}$, provides a lower bound of the variance - covariance matrix for unbiased estimators of $\theta$ .

$$I(\theta)^{-1} = \begin{pmatrix} \sigma^2(X'X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}$$

For large $n$, we approximately obtain: $\begin{pmatrix} \tilde{\beta} \\ \tilde{\sigma}^2 \end{pmatrix} \sim N\left(\begin{pmatrix} \beta \\ \sigma^2 \end{pmatrix}, \begin{pmatrix} \sigma^2(X'X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}\right)$.

## 9.3 MLE: The Case of Multiple Regression Model II

1. Regression model: $\quad y = X\beta + u, \quad u \sim N(0, \sigma^2\Omega)$

   Transformation of Variables from $u$ to $y$:

$$f_u(u) = (2\pi\sigma^2)^{-n/2}|\Omega|^{-1/2}\exp\Big(-\frac{1}{2\sigma^2}u'\Omega^{-1}u\Big)$$

$$\begin{aligned} f_y(y) &= f_u(y - X\beta)\left|\frac{\partial u}{\partial y'}\right| \\ &= (2\pi\sigma^2)^{-n/2}|\Omega|^{-1/2}\exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)'\Omega^{-1}(y - X\beta)\Big) \\ &= L(\theta; y, X), \end{aligned}$$

   where $\theta = (\beta, \sigma^2)$, because of $\dfrac{\partial u}{\partial y'} = I_n$.

   The log-likelihood function is:

$$\log L(\theta; y, X) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\log|\Omega| - \frac{1}{2\sigma^2}(y - X\beta)'\Omega^{-1}(y - X\beta),$$

where $\theta = (\beta, \sigma^2)$.

2. $\max_{\theta} \quad \log L(\theta; y, X)$

$$\text{(FOC)} \quad \frac{\partial \log L(\theta; y, X)}{\partial \theta} = 0$$

$$\text{(SOC)} \quad \frac{\partial^2 \log L(\theta; y, X)}{\partial \theta \partial \theta'} \text{ is a negative definite matrix.}$$

Then, we obtain MLE of $\beta$ and $\sigma^2$:

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y, \qquad \tilde{\sigma}^2 = \frac{(y - X\tilde{\beta})'\Omega^{-1}(y - X\tilde{\beta})}{n}$$

3. Fisher's information matrix is defined as:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; y, X)}{\partial \theta \partial \theta'}\Big)$$

The inverse of the information matrix, $I(\theta)^{-1}$, provides a lower bound of the variance - covariance matrix for unbiased estimators of $\theta$, which is given by:

$$I(\theta)^{-1} = \begin{pmatrix} \sigma^2(X'\Omega^{-1}X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}$$

## 9.4 MLE: AR(1) Model

The $p$th-order Autoregressive Model, i.e., AR($p$) Model ($p$ 次の自己回帰モデル):

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + u_t$$

AR(1) Model: $\quad t = 2, 3, \cdots, n,$

$$y_t = \phi_1 y_{t-1} + u_t, \qquad u_t \sim N(0, \sigma^2)$$

where $|\phi_1| < 1$ is assumed for now.

To obtain the joint density function of $y_1, y_2, \cdots, y_n$, $f(y_n, y_{n-1}, \cdots, y_1)$ is decomposed as follows:

$$f(y_n, y_{n-1}, \cdots, y_1) = f(y_1) \prod_{t=2}^{n} f(y_t | y_{t-1}, \cdots, y_1).$$

From $y_t = \phi_1 y_{t-1} + u_t$, we can obtain:

$$\mathrm{E}(y_t|y_{t-1},\cdots,y_1) = \phi_1 y_{t-1}, \qquad \text{and} \qquad \mathrm{V}(y_t|y_{t-1},\cdots,y_1) = \sigma^2.$$

Therefore, the conditional distribution $f(y_t|y_{t-1},\cdots,y_1)$ is:

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right).$$

To obtain the unconditional distribution $f(y_t)$, $y_t$ is rewritten as follows:

$$\begin{aligned}
y_t &= \phi_1 y_{t-1} + u_t \\
&= \phi_1^2 y_{t-2} + u_t + \phi_1 u_{t-1} \\
&\;\;\vdots \\
&= \phi_1^j y_{t-j} + u_t + \phi_1 u_{t-1} + \cdots + \phi_1^j u_{t-j} \\
&\;\;\vdots \\
&= u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots, \qquad \text{when } j \text{ goes to infinity.}
\end{aligned}$$

The unconditional expectation and variance of $y_t$ is:

$$\mathrm{E}(y_t) = 0, \quad \text{and} \quad \mathrm{V}(y_t) = \sigma^2(1 + \phi_1^2 + \phi_1^4 + \cdots) = \frac{\sigma^2}{1 - \phi_1^2}.$$

Therefore, the unconditional distribution of $y_t$ is given by:

$$f(y_t) = \frac{1}{\sqrt{2\pi\sigma^2/(1 - \phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1 - \phi_1^2)} y_t^2\right).$$

Finally, the joint distribution of $y_1, y_2, \cdots, y_n$ is given by:

$$
\begin{aligned}
f(y_n, y_{n-1}, \cdots, y_1) &= f(y_1) \prod_{t=2}^{n} f(y_t | y_{t-1}, \cdots, y_1) \\
&= \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)} y_1^2\right) \\
&\qquad \times \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)
\end{aligned}
$$

The log-likelihood function is:

$$\begin{aligned}\log L(\phi_1, \sigma^2; y_n, y_{n-1}, \cdots, y_1) = &-\frac{1}{2}\log(2\pi\sigma^2/(1-\phi_1^2)) - \frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 \\ &-\frac{n-1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(y_t - \phi_1 y_{t-1})^2.\end{aligned}$$

Maximize $\log L$ with respect to $\phi_1$ and $\sigma^2$.

Maximization Procedure:

- Newton-Raphson Method, or Method of Scoring
- Simple Grid Search (search maximization within the range $-1 < \phi_1 < 1$, changing the value of $\phi_1$ by 0.01)

## 9.5 MLE: Regression Model with AR(1) Error

When the error term is autocorrelated, the regression model is written as:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2).$$

The joint distribution of $u_n, u_{n-1}, \cdots, u_1$ is:

$$\begin{aligned}
f_u(u_n, u_{n-1}, \cdots, u_1; \rho, \sigma_\epsilon^2) &= f_u(u_1; \rho, \sigma_\epsilon^2) \prod_{t=2}^{n} f_u(u_t | u_{t-1}, \cdots, u_1; \rho, \sigma_\epsilon^2) \\
&= (2\pi\sigma_\epsilon^2/(1-\rho^2))^{-1/2} \exp\left(-\frac{1}{2\sigma_\epsilon^2/(1-\rho^2)} u_1^2\right) \\
&\qquad \times (2\pi\sigma_\epsilon^2)^{-(n-1)/2} \exp\left(-\frac{1}{2\sigma_\epsilon^2} \sum_{t=2}^{n} (u_t - \rho u_{t-1})^2\right).
\end{aligned}$$

By transformation of variables from $u_n, u_{n-1}, \cdots, u_1$ to $y_n, y_{n-1}, \cdots, y_1$, the joint distribution of $y_n, y_{n-1}, \cdots, y_1$ is:

$$
\begin{aligned}
& f_y(y_n, y_{n-1}, \cdots, y_1; \rho, \sigma_\epsilon^2, \beta) \\
&\quad = f_u(y_n - x_n\beta, y_{n-1} - x_{n-1}\beta, \cdots, y_1 - x_1\beta; \rho, \sigma_\epsilon^2) \left| \frac{\partial u}{\partial y'} \right| \\
&\quad = (2\pi\sigma_\epsilon^2/(1-\rho^2))^{-1/2} \exp\left( -\frac{1}{2\sigma_\epsilon^2/(1-\rho^2)} (y_1 - x_1\beta)^2 \right) \\
&\qquad\qquad \times (2\pi\sigma_\epsilon^2)^{-(n-1)/2} \exp\left( -\frac{1}{2\sigma_\epsilon^2} \sum_{t=2}^{n} \Big( (y_t - \rho y_{t-1}) - (x_t - \rho x_{t-1})\beta \Big)^2 \right) \\
&\quad = (2\pi\sigma_\epsilon^2)^{-1/2} (1-\rho^2)^{1/2} \exp\left( -\frac{1}{2\sigma_\epsilon^2} ( \sqrt{1-\rho^2} y_1 - \sqrt{1-\rho^2} x_1\beta)^2 \right) \\
&\qquad\qquad \times (2\pi\sigma_\epsilon^2)^{-(n-1)/2} \exp\left( -\frac{1}{2\sigma_\epsilon^2} \sum_{t=2}^{n} \Big( (y_t - \rho y_{t-1}) - (x_t - \rho x_{t-1})\beta \Big)^2 \right) \\
&\quad = (2\pi\sigma_\epsilon^2)^{-n/2} (1-\rho^2)^{1/2} \exp\left( -\frac{1}{2\sigma_\epsilon^2} (y_1^* - x_1^*\beta)^2 \right) \times \exp\left( -\frac{1}{2\sigma_\epsilon^2} \sum_{t=2}^{n} (y_t^* - x_t^*\beta)^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= (2\pi)^{-n/2}(\sigma_\epsilon^2)^{-n/2}(1-\rho^2)^{1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=1}^{n}(y_t^* - x_t^*\beta)^2\right) \\
&= L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1),
\end{aligned}
$$

where $y_t^*$ and $x_t^*$ are given by:

$$
y_t^* = \begin{cases} \sqrt{1-\rho^2}\,y_t, & \text{for } t = 1, \\ y_t - \rho y_{t-1}, & \text{for } t = 2, 3, \cdots, n, \end{cases}
$$

$$
x_t^* = \begin{cases} \sqrt{1-\rho^2}\,x_t, & \text{for } t = 1, \\ x_t - \rho x_{t-1}, & \text{for } t = 2, 3, \cdots, n, \end{cases}
$$

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to $\beta$ should be zero.

$$\begin{aligned}\tilde{\beta} &= (\sum_{t=1}^{T} x_t^{*\prime} x_t^*)^{-1} (\sum_{t=1}^{T} x_t^{*\prime} y_t^*) \\ &= (X^{*\prime} X^*)^{-1} X^{*\prime} y^*\end{aligned}$$

$\Longrightarrow$ This is equivalent to OLS from the regression model: $y^* = X^*\beta + \epsilon$ and $\epsilon \sim N(0, \sigma^2 I_n)$, where $\sigma^2 = \sigma_\epsilon^2/(1-\rho^2)$.

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to $\sigma_\epsilon^2$ should be zero.

$$\tilde{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t^* - x_t^*\beta)^2 = \frac{1}{n}(y^* - X^*\beta)'(y^* - X^*\beta),$$

where

$$y^* = \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_n - \rho y_{n-1} \end{pmatrix}, \qquad X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}x_1 \\ x_2 - \rho x_1 \\ \vdots \\ x_n - \rho x_{n-1} \end{pmatrix}.$$

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to $\rho$ should be zero.

$$\max_{\beta,\sigma_\epsilon^2,\rho} L(\rho, \sigma_\epsilon^2, \beta; y) \quad \text{is equivalent to} \quad \max_{\rho} L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y).$$

$L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y)$ is called the **concentrated log-likelihood function** (集約対数尤度関数), which is a function of $\rho$, i.e., both $\tilde{\sigma}_\epsilon^2$ and $\tilde{\beta}$ depend only on $\rho$.

The log-likelihood function is written as:

$$\begin{aligned}
\log L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y) &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\tilde{\sigma}_\epsilon^2) + \frac{1}{2}\log(1-\rho^2) - \frac{n}{2} \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2} - \frac{n}{2}\log\Big(\tilde{\sigma}_\epsilon^2(\rho)\Big) + \frac{1}{2}\log(1-\rho^2)
\end{aligned}$$

For maximization of $\log L$, use Newton-Raphson method, method of scoring or simple grid search

Note that $\tilde{\sigma}_\epsilon^2 = \tilde{\sigma}_\epsilon^2(\rho) = \dfrac{1}{n}(y^* - X^*\tilde{\beta})'(y^* - X^*\tilde{\beta})$ for $\tilde{\beta} = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*$.

**Remark:** The regression model with AR(1) error is:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2).$$

$$\mathrm{V}(u) = \sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ \rho^3 & \rho^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \rho \\ \rho^{n-1} & \rho^{n-2} & \cdots & \rho^2 & \rho & 1 \end{pmatrix} = \sigma^2\Omega, \qquad \text{where } \sigma^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}.$$

where $\mathrm{Cov}(u_i, u_j) = \mathrm{E}(u_i u_j) = \sigma^2\rho^{|i-j|}$, i.e., the $i$th row and $j$th column of $\Omega$ is $\rho^{|i-j|}$.

The regression model with AR(1) error is: $\qquad y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$

There exists $P$ which satisfies that $\Omega = PP'$, because $\Omega$ is a positive definite matrix.

Multiply $P^{-1}$ on both sides from the left.

$$
\begin{aligned}
P^{-1}y = P^{-1}X\beta + P^{-1}u \qquad &\Longrightarrow \qquad y^* = X^*\beta + u^* \text{ and } u^* \sim N(0, \sigma^2 I_n) \\
&\Longrightarrow \qquad \text{Apply OLS.}
\end{aligned}
$$

$$
y^* = \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}\, y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_n - \rho y_{n-1} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \cdots & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = P^{-1} y
$$

$$
X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}\, x_1 \\ x_2 - \rho x_1 \\ \vdots \\ x_n - \rho x_{n-1} \end{pmatrix} = P^{-1} X \qquad \Longrightarrow
$$

Check $P^{-1}\Omega P^{-1\prime} = aI_n$, where $a$ is constant.

## 9.6 MLE: Regression Model with Heteroscedastic Errors

In the case where the error term depends on the other exogenous variables, the regression model is written as follows:

$$y_i = x_i\beta + u_i, \qquad u_i \sim \text{ id } N(0, \sigma_i^2), \qquad \sigma_i^2 = (z_i\alpha)^2.$$

The joint distribution of $u_n, u_{n-1}, \cdots, u_1$, denoted by $f_u(\cdot;\cdot)$, is given by:

$$\begin{aligned}
\log f_u(u_n, u_{n-1}, \cdots, u_1; \sigma_1^2, \cdots, \sigma_n^2) &= \sum_{i=1}^{n} \log f_u(u_t; \sigma_i^2) \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log(\sigma_i^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i}{\sigma_i}\right)^2 \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i}{z_i\alpha}\right)^2
\end{aligned}$$

By the transformation of variables from $u_n, u_{n-1}, \cdots, u_1$ to $y_n, y_{n-1}, \cdots, y_1$, the log-likelihood function is:

$$
\begin{aligned}
L(\alpha,\beta;y_n,y_{n-1},\cdots,y_1) &= \log f_y(y_n,y_{n-1},\cdots,y_1;\alpha,\beta) \\
&= \log f_u(y_n - x_n\beta, y_{n-1} - x_{n-1}\beta,\cdots,y_1 - x_1\beta;\sigma_i^2)\left|\frac{\partial u}{\partial y'}\right| \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i - x_i\beta}{z_i\alpha}\right)^2
\end{aligned}
$$

$\Longrightarrow$ Maximize the above log-likelihood function with respect to $\beta$ and $\alpha$.

# 10 Asymptotic Theory

1. **Definition: Convergence in Distribution (分布収束)**

   A series of random variables $X_1$, $X_2$, $\cdots$, $X_n$, $\cdots$ have distribution functions $F_1$, $F_2$, $\cdots$, respectively.

   If

   $$\lim_{n\to\infty} F_n = F,$$

   then we say that a series of random variables $X_1$, $X_2$, $\cdots$ converges to $F$ in distribution.

2. **Consistency (一致性):**

   (a) **Definition: Convergence in Probability (確率収束)**

   Let $\{Z_n :\ n = 1, 2, \cdots\}$ be a series of random variables.

If the following holds,

$$\lim_{n\to\infty} P(|Z_n - \theta| < \epsilon) = 1,$$

for any positive $\epsilon$, then we say that $Z_n$ converges to $\theta$ in probability.

$\theta$ is called a **probability limit (確率極限)** of $Z_n$.

$$\text{plim } Z_n = \theta.$$

(b) Let $\hat{\theta}_n$ be an estimator of parameter $\theta$.

If $\hat{\theta}_n$ converges to $\theta$ in probability, we say that $\hat{\theta}_n$ is a consistent estimator of $\theta$.

3. A General Case of **Chebyshev's Inequality:**

For $g(X) \geq 0$,

$$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

where $k$ is a positive constant.

4. **Example:** For a random variable $X$, set $g(X) = (X - \mu)'(X - \mu)$, $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \Sigma$.

   Then, we have the following inequality:

$$P((X - \mu)'(X - \mu) \geq k) \leq \frac{\mathrm{tr}(\Sigma)}{k}.$$

   Note as follows:

$$\begin{aligned}\mathrm{E}((X - \mu)'(X - \mu)) &= \mathrm{E}\Big(\mathrm{tr}((X - \mu)'(X - \mu))\Big) = \mathrm{E}\Big(\mathrm{tr}((X - \mu)(X - \mu)')\Big) \\ &= \mathrm{tr}\Big(\mathrm{E}((X - \mu)(X - \mu)')\Big) = \mathrm{tr}(\Sigma).\end{aligned}$$

5. **Example 1 (Univariate Case):**

Suppose that $X_i \sim (\mu, \sigma^2)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of $\mu$.

**Proof:**

Note that $g(\overline{X}) = (\overline{X} - \mu)^2$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{V}(\overline{X}) = \dfrac{\sigma^2}{n}$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0, \qquad \text{for any } \epsilon.$$

That is. for any $\epsilon$,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| < \epsilon) = 1.$$

$\Longrightarrow$ **Chebyshev's inequality**

6. **Example 2 (Multivariate Case):**

Suppose that $X_i \sim (\mu, \Sigma)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of $\mu$.

**Proof:**

Note that $g(\overline{X}) = (\overline{X} - \mu)'(\overline{X} - \mu)$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{tr}\Big(\mathrm{V}(\overline{X})\Big) = \mathrm{tr}\Big(\frac{1}{n}\Sigma\Big)$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P((\overline{X} - \mu)'(\overline{X} - \mu) \geq k) = P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\mathrm{tr}(\Sigma)}{n\epsilon^2} \longrightarrow 0, \quad \text{for any positive } \epsilon.$$

That is. for any positive $\epsilon$, $\lim_{n\to\infty} P((\overline{X} - \mu)'(\overline{X} - \mu) < k) = 1$.

Note that $|\overline{X} - \mu| = \sqrt{(\overline{X} - \mu)'(\overline{X} - \mu)}$, which is the distance between $X$ and $\mu$.

$\Longrightarrow$ **Chebyshev's inequality**

7. **Some Formulas:**

Let $X_n$ and $Y_n$ be the random variables which satisfy $\operatorname{plim} X_n = c$ and $\operatorname{plim} Y_n = d$. Then,

(a) $\operatorname{plim} (X_n + Y_n) = c + d$

(b) $\operatorname{plim} X_n Y_n = cd$

(c) $\operatorname{plim} X_n / Y_n = c/d$ for $d \neq 0$

(d) $\operatorname{plim} g(X_n) = g(c)$ for a function $g(\cdot)$

$\Longrightarrow$ **Slutsky's Theorem (スルツキー定理)**

8. **Central Limit Theorem (中心極限定理)**

**Univariate Case:** $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \sigma^2)$.

Then,

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \longrightarrow N(0, 1),$$

which implies

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \sigma^2).$$

**Multivariate Case:** $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma)$$

9. **Central Limit Theorem (Generalization)**

$X_1,\ X_2,\ \cdots,\ X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma_i)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma),$$

where

$$\Sigma = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \Sigma_i \right).$$

10. **Definition:** Let $\hat{\theta}_n$ be a consistent estimator of $\theta$.

Suppose that $\sqrt{n}(\hat{\theta}_n - \theta)$ converges to $N(0, \Sigma)$ in distribution.

Then, we say that $\hat{\theta}_n$ has an **asymptotic distribution (漸近分布)**: $N(\theta, \Sigma/n)$.

11. $X_1, X_2, \cdots, X_n$ are random variables with density function $f(x;\theta)$.

Let $\hat{\theta}_n$ be a maximum likelihood estimator of $\theta$.

Then, under some **regularity conditions**. $\hat{\theta}_n$ is a consistent estimator of $\theta$ and the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is given by: $N\left(0, \lim\left(\dfrac{I(\theta)}{n}\right)^{-1}\right)$.

12. **Regularity Conditions:**

  (a) The domain of $X_i$ does not depend on $\theta$.

  (b) There exists at least third-order derivative of $f(x;\theta)$ with respect to $\theta$, and their derivatives are finite.

13. Thus, MLE is

(i) consistent,

(ii) asymptotically normal, and

(iii) asymptotically efficient.

# 11 Consistency and Asymptotic Normality of OLSE

Regression model: $\qquad y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n).$

**Consistency:**

1. Let $\hat{\beta}_n = (X'X)^{-1}X'y$ be the OLS with sample size $n$.

   Consistency: As $n$ is large, $\hat{\beta}_n$ converges to $\beta$.

2. Assume the stationarity assumption for $X$, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

   Then, we have the following result:

$$\frac{1}{n}X'u \longrightarrow 0.$$

**Proof:**

According to Chebyshev's inequality, for $g(Z) \geq 0$,

$$P(g(Z) \geq k) \leq \frac{\mathrm{E}(g(Z))}{k},$$

where $k$ is a positive constant.

Set $g(Z) = Z'Z$, and $Z = \dfrac{1}{n}X'u$.

Apply Chebyshev's inequality.

$$\begin{aligned}\mathrm{E}\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u\Big) &= \frac{1}{n^2}\mathrm{E}\Big(u'XX'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(u'XX'u)\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(XX'uu')\Big) \\ &= \frac{1}{n^2}\mathrm{tr}\Big(XX'\mathrm{E}(uu')\Big) = \frac{\sigma^2}{n^2}\mathrm{tr}(XX') = \frac{\sigma^2}{n^2}\mathrm{tr}(X'X) = \frac{\sigma^2}{n}\mathrm{tr}(\frac{1}{n}X'X).\end{aligned}$$

Therefore,

$$P\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u \geq k\Big) \leq \frac{\sigma^2}{nk}\mathrm{tr}(\frac{1}{n}X'X) \longrightarrow 0 \times \mathrm{tr}(M_{xx}) = 0.$$

Note that from the assumption,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Therefore, we have:

$$(\frac{1}{n}X'u)'\frac{1}{n}X'u \longrightarrow 0,$$

which implies:

$$\frac{1}{n}X'u \longrightarrow 0,$$

because $(\frac{1}{n}X'u)'\frac{1}{n}X'u$ indicates a quadratic form.

3. Note that $\frac{1}{n}X'X \longrightarrow M_{xx}$ results in $(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$.

$\Longrightarrow$ Slutsky's Theorem

(*) **Slutsky's Theorem** $g(\hat{\theta}) \longrightarrow g(\theta)$, when $\hat{\theta} \longrightarrow \theta$.

4. OLS is given by:

$$\hat{\beta}_n = \beta + (X'X)^{-1}X'u = \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat{\beta}_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consitent estimator.

**Asymptotic Normality:**

1. Asymptotic Normality of OLSE

$$\sqrt{n}(\hat{\beta}_n - \beta) \;\longrightarrow\; N(0.\sigma^2 M_{xx}^{-1}), \quad \text{when } n \;\longrightarrow\; \infty.$$

2. **Central Limit Theorem:** Greenberg and Webster (1983)

   $Z_1, Z_2, \cdots, Z_n$ are mutually indelendently distributed with mean $\mu$ and variance $\Sigma_i$.

   Then, we have the following result:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i - \mu) \longrightarrow N(0, \Sigma),$$

   where

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\Sigma_i\right).$$

   The distribution of $Z_i$ is not assumed.

3. Define $Z_i = x_i' u_i$. Then, $\Sigma_i = \mathrm{Var}(Z_i) = \sigma^2 x_i' x_i$.

4. $\Sigma$ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$
\begin{aligned}
\mathrm{Var}\Big(\Big(\frac{1}{n}X'X\Big)^{-1}\frac{1}{\sqrt{n}}X'u\Big) &= \mathrm{E}\Big(\Big(\frac{1}{n}X'X\Big)^{-1}\frac{1}{\sqrt{n}}X'u\Big(\Big(\frac{1}{n}X'X\Big)^{-1}\frac{1}{\sqrt{n}}X'u\Big)'\Big) \\
&= \Big(\frac{1}{n}X'X\Big)^{-1}\Big(\frac{1}{n}X'\mathrm{E}(uu')X\Big)\Big(\frac{1}{n}X'X\Big)^{-1} \\
&= \sigma^2\Big(\frac{1}{n}X'X\Big)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}.
\end{aligned}
$$

Therefore,

$$
\sqrt{n}(\hat{\beta}-\beta) \longrightarrow N(0, \sigma^2 M_{xx}^{-1})
$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of $u_i$ is not assumed.

# 12 Instrumental Variable (操作変数法)

## 12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

   $V$: is called the **measurement error (測定誤差 or 観測誤差)**.

3. For the elements which do not include measurement errors in $X$, the corresponding elements in $V$ are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of $\beta$ is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

   (a) The measurement error in $X$ is uncorrelated with $\tilde{X}$ in the limit. i.e.,

   $$\text{plim}\Big(\frac{1}{n}\tilde{X}'V\Big) = 0.$$

   Therefore, we obtain the following:

   $$\text{plim}\Big(\frac{1}{n}X'X\Big) = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) + \text{plim}\Big(\frac{1}{n}V'V\Big) = \Sigma + \Omega$$

(b) $u$ is not correlated with $V$.

$u$ is not correlated with $\tilde{X}$.

That is,

$$\operatorname{plim}\Big(\frac{1}{n}V'u\Big) = 0, \qquad \operatorname{plim}\Big(\frac{1}{n}\tilde{X}'u\Big) = 0.$$

6. OLSE of $\beta$ is:

$$\hat{\beta} = \beta + (X'X)^{-1}X'(u - V\beta) = \beta + (X'X)^{-1}(\tilde{X} + V)'(u - V\beta).$$

Therefore, we obtain the following:

$$\operatorname{plim}\hat{\beta} = \beta - (\Sigma + \Omega)^{-1}\Omega\beta$$

7. **Example: The Case of Two Variables:**

The regression model is given by:

$$y_t = \alpha + \beta \tilde{x}_t + u_t, \qquad x_t = \tilde{x}_t + v_t.$$

Under the above model,

$$\Sigma = \mathrm{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) = \mathrm{plim}\begin{pmatrix} 1 & \dfrac{1}{n}\sum \tilde{x}_i \\ \dfrac{1}{n}\sum \tilde{x}_i & \dfrac{1}{n}\sum \tilde{x}_i^2 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix},$$

where $\mu$ and $\sigma^2$ represent the mean and variance of $\tilde{x}_i$.

$$\Omega = \mathrm{plim}\Big(\frac{1}{n}V'V\Big) = \mathrm{plim}\begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{n}\sum v_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\mathrm{plim}\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \left(\begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \frac{1}{\sigma^2 + \sigma_v^2}\begin{pmatrix} -\mu\sigma_v^2\beta \\ \sigma_v^2\beta \end{pmatrix}
\end{aligned}$$

Now we focus on $\beta$.

$\hat{\beta}$ is not consistent. because of:

$$\mathrm{plim}(\hat{\beta}) = \beta - \frac{\sigma_v^2 \beta}{\sigma^2 + \sigma_v^2} = \frac{\beta}{1 + \sigma_v^2/\sigma^2} < \beta$$

## 12.2 Instrumental Variable (IV) Method (操作変数法 or IV 法)

Instrumental Variable (IV)

1. Consider the regression model: $y = X\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE of $\beta$ is inconsistent.

2. **Proof:**

$$\hat{\beta} = \beta + (\frac{1}{n}X'X)^{-1}\frac{1}{n}X'u \longrightarrow \beta + M_{xx}^{-1}M_{xu},$$

   where

$$\frac{1}{n}X'X \longrightarrow M_{xx}, \qquad \frac{1}{n}X'u \longrightarrow M_{xu} \neq 0$$

3. Find the $Z$ which satisfies $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

Multiplying $Z'$ on both sides of the regression model: $y = X\beta + u$,

$$Z'y = Z'X\beta + Z'u$$

Dividing $n$ on both sides of the above equation, we take plim on both sides.

Then, we obtain the following:

$$\text{plim}\left(\frac{1}{n}Z'y\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta + \text{plim}\left(\frac{1}{n}Z'u\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta.$$

Accordingly, we obtain:

$$\beta = \left(\text{plim}\left(\frac{1}{n}Z'X\right)\right)^{-1} \text{plim}\left(\frac{1}{n}Z'y\right).$$

Therefore, we consider the following estimator:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

which is taken as an estimator of $\beta$.

$\Longrightarrow$ **Instrumental Variable Method (操作変数法 or IV 法)**

4. Assume the followings:

$$\frac{1}{n}Z'X \longrightarrow M_{zx}, \qquad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \qquad \frac{1}{n}Z'u \longrightarrow 0$$

5. **Asymptotic Distribution of $\beta_{IV}$:**

$$\beta_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u,$$

which is rewritten as:

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big)$$

Applying the Central Limit Theorem to $\Big(\frac{1}{\sqrt{n}}Z'u\Big)$, we have the following result:

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Therefore,

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}'^{-1})$$

$\Longrightarrow$ Consistency and Asymptotic Normality

6. The variance of $\beta_{IV}$ is given by:

$$\mathrm{V}(\beta_{IV}) = s^2 (Z'X)^{-1} Z'Z (X'Z)^{-1},$$

where

$$s^2 = \frac{(y - X\beta_{IV})'(y - X\beta_{IV})}{n - k}.$$

## 12.3 Two-Stage Least Squares Method (2 段階最小二乗法, 2SLS or TSLS)

1. Regression Model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I),$$

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE is not consistent.

2. Find the variable $Z$ which satisfies $\dfrac{1}{n} Z'u \;\longrightarrow\; M_{zu} = 0$.

3. Use $Z = \hat{X}$ for the instrumental variable.

   $\hat{X}$ is the predicted value which regresses $X$ on the other exogenous variables, say $W$.

   That is, consider the following regression model:

$$X = WB + V.$$

Estimate $B$ by OLS.

Then, we obtain the prediction:

$$\hat{X} = W\hat{B},$$

where $\hat{B} = (W'W)^{-1}W'X$.

Or, equivalently,

$$\hat{X} = W(W'W)^{-1}W'X.$$

$\hat{X}$ is used for the instrumental variable of $X$.

4. The IV method is rewritten as:

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'y.$$

Furthermore, $\beta_{IV}$ is written as follows:

$$\beta_{IV} = \beta + (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'u.$$

Therefore, we obtain the following expression:

$$\sqrt{n}(\beta_{IV}-\beta) = \left(\left(\frac{1}{n}X'W\right)\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{1}{n}XW'\right)'\right)^{-1}\left(\frac{1}{n}X'W\right)\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{1}{\sqrt{n}}W'u\right)$$
$$\longrightarrow N\Big(0, \sigma^2(M_{xw}M_{ww}^{-1}M'_{xw})^{-1}\Big).$$

5. Clearly, there is no correlation between $W$ and $u$ at least in the limit, i.e.,

$$\operatorname{plim}\left(\frac{1}{n}W'u\right) = 0.$$

6. **Remark:**

$$\hat{X}'X = X'W(W'W)^{-1}W'X = X'W(W'W)^{-1}W'W(W'W)^{-1}W'X = \hat{X}'\hat{X}.$$

Therefore,

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (\hat{X}'\hat{X})^{-1}\hat{X}'y,$$

which implies the OLS estimator of $\beta$ in the regression model: $y = \hat{X}\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

**Example:**

$$y_t = \alpha x_t + \beta z_t + u_t, \qquad u_t \sim (0, \sigma^2).$$

Suppose that $x_t$ is correlated with $u_t$ but $z_t$ is not correlated with $u_t$.

- 1st Step:

Estimate the following regression model:

$$x_t = \gamma w_t + \delta z_t + \cdots + v_t,$$

by OLS. $\Longrightarrow$ Obtain $\hat{x}_t$ through OLS.

- 2nd Step:

Estimate the following regression model:

$$y_t = \alpha \hat{x}_t + \beta z_t + u_t,$$

by OLS. $\Longrightarrow$ $\alpha_{iv}$ and $\beta_{iv}$

Note as follows. Estimate the following regression model:

$$z_t = \gamma_2 w_t + \delta_2 z_t + \cdots + v_{2t},$$

by OLS.

$\Longrightarrow$ $\hat{\gamma}_2 = 0$, $\hat{\delta}_2 = 1$, and the other coefficient estimates are zeros. i.e., $\hat{z}_t = z_t$.

Eviews Command:

```
tsls y x z @ w z ...
```

# 13 Large Sample Tests

## 13.1 Wald, LM and LR Tests

Parameter $\theta : k \times 1$, $\qquad h(\theta) : G \times 1$ vector function, $G \leq k$

The null hypothesis $H_0 : h(\theta) = 0 \implies G$ restrictions

$\tilde{\theta} : k \times 1$, restricted maximum likelihood estimate

$\hat{\theta} : k \times 1$, unrestricted maximum likelihood estimate

$I(\theta) : k \times k$, information matrix, i.e., $\quad I(\theta) = -\mathrm{E}\Big(\dfrac{\partial^2 \log L(\theta)}{\partial\theta\partial\theta'}\Big)$.

$\log L(\theta)$ : log-likelihood function

$R_\theta = \dfrac{\partial h(\theta)}{\partial\theta'} : G \times k, \qquad F_\theta = \dfrac{\partial \log L(\theta)}{\partial\theta} : k \times 1$

1. **Wald Test (ワルド検定):** $\quad W = h(\hat{\theta})'\Big(R_{\hat{\theta}}(I(\hat{\theta}))^{-1}R'_{\hat{\theta}}\Big)^{-1}h(\hat{\theta})$

   (a) $h(\hat{\theta}) \approx h(\theta) + \dfrac{\partial h(\theta)}{\partial\theta'}(\hat{\theta} - \theta) \quad \Longleftarrow \quad h(\hat{\theta})$ is linearized around $\hat{\theta} = \theta$.

Under the null hypothesis $h(\theta) = 0$,

$$h(\hat{\theta}) \approx \frac{\partial h(\theta)}{\partial \theta'}(\hat{\theta} - \theta) = R_\theta(\hat{\theta} - \theta)$$

(b) $\hat{\theta}$ is MLE.

From the properties of MLE,

$$\sqrt{n}(\hat{\theta} - \theta) \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\frac{I(\theta)}{n}\Big)^{-1}\Big),$$

That is, approximately, we have the following result:

$$\hat{\theta} - \theta \sim N\Big(0, (I(\theta))^{-1}\Big).$$

(c) The distribution of $h(\hat{\theta})$ is approximately given by:

$$h(\hat{\theta}) \sim N\Big(0, R_\theta (I(\theta))^{-1} R'_\theta\Big)$$

(d) Therefore, the $\chi^2(G)$ distribution is derived as follows:

$$h(\hat{\theta})\Big(R_\theta(I(\theta))^{-1}R'_\theta\Big)^{-1}h(\hat{\theta})' \longrightarrow \chi^2(G).$$

Furthermore, from the fact that $R_{\hat{\theta}} \longrightarrow R_\theta$ and $I(\hat{\theta}) \longrightarrow I(\theta)$ as $n \longrightarrow \infty$ (i.e., convergence in probability, 確率収束), we can replace $\theta$ by $\hat{\theta}$ as follows:

$$h(\hat{\theta})\Big(R_{\hat{\theta}}(I(\hat{\theta}))^{-1}R'_{\hat{\theta}}\Big)^{-1}h(\hat{\theta})' \longrightarrow \chi^2(G).$$

2. **Lagrange Multiplier Test (ラグランジェ乗数検定):** $LM = F'_{\tilde{\theta}}(I(\tilde{\theta}))^{-1}F_{\tilde{\theta}}$

(a) MLE with the constraint $h(\theta) = 0$:

$$\max_{\theta} \ \log L(\theta), \quad \text{subject to} \quad h(\theta) = 0$$

The Lagrangian function is: $L = \log L(\theta) + \lambda h(\theta)$.

(b) For maximization, we have the following two equations:

$$\frac{\partial L}{\partial \theta} = \frac{\partial \log L(\theta)}{\partial \theta} + \lambda \frac{\partial h(\theta)}{\partial \theta} = 0, \qquad \frac{\partial L}{\partial \lambda} = h(\theta) = 0.$$

The restricted MLE $\tilde{\theta}$ satisfies $h(\tilde{\theta}) = 0$.

(c) Mean and variance of $\dfrac{\partial \log L(\theta)}{\partial \theta}$ are given by:

$$\mathrm{E}\Big(\frac{\partial \log L(\theta)}{\partial \theta}\Big) = 0, \qquad \mathrm{V}\Big(\frac{\partial \log L(\theta)}{\partial \theta}\Big) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}\Big) = I(\theta).$$

(d) Therefore, using the central limit theorem,

$$\frac{1}{\sqrt{n}} \frac{\partial \log L(\theta)}{\partial \theta} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log f(X_i; \theta)}{\partial \theta} \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\frac{1}{n} I(\theta)\Big)\Big)$$

(e) Therefore, $\dfrac{\partial \log L(\theta)}{\partial \theta} (I(\theta))^{-1} \dfrac{\partial \log L(\theta)}{\partial \theta'} \longrightarrow \chi^2(G)$.

Under $H_0:\ h(\theta) = 0$, replacing $\theta$ by $\tilde{\theta}$ we have the result:

$$F'_{\tilde{\theta}} (I(\tilde{\theta}))^{-1} F_{\tilde{\theta}} \longrightarrow \chi^2(G).$$

3. **Likelihood Ratio Test (尤度比検定):** $LR = -2\log\lambda \longrightarrow \chi^2(G)$

$$\lambda = \frac{L(\tilde{\theta})}{L(\hat{\theta})}$$

(a) By Taylor series expansion evaluated at $\theta = \hat{\theta}$, $\log L(\theta)$ is given by:

$$\begin{aligned}\log L(\theta) &= \log L(\hat{\theta}) + \frac{\partial \log L(\hat{\theta})}{\partial\theta}(\theta - \hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})'\frac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}(\theta - \hat{\theta}) + \cdots \\ &= \log L(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})'\frac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}(\theta - \hat{\theta}) + \cdots\end{aligned}$$

Note that $\dfrac{\partial \log L(\hat{\theta})}{\partial\theta} = 0$ because $\hat{\theta}$ is MLE.

$$\begin{aligned}-2(\log L(\theta) - \log L(\hat{\theta})) &\approx -(\theta - \hat{\theta})'\Big(\frac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}\Big)(\theta - \hat{\theta}) \\ &= \sqrt{n}(\hat{\theta} - \theta)'\Big(-\frac{1}{n}\frac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}\Big)\sqrt{n}(\hat{\theta} - \theta) \\ &\longrightarrow \chi^2(G)\end{aligned}$$

Note:

(1) $\hat{\theta} \longrightarrow \theta$,

(2) $-\dfrac{1}{n}\dfrac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'} \longrightarrow -\lim_{n\to\infty}\Big(\dfrac{1}{n}\mathrm{E}\Big(\dfrac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}\Big)\Big) = \lim_{n\to\infty}\Big(\dfrac{1}{n}I(\theta)\Big)$,

(3) $\sqrt{n}(\hat{\theta}-\theta) \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\dfrac{1}{n}I(\theta)\Big)\Big)$.

(b) Under $H_0 : \; h(\theta) = 0$,

$$-2(\log L(\tilde{\theta}) - \log L(\hat{\theta})) \longrightarrow \chi^2(G).$$

Remember that $h(\tilde{\theta}) = 0$ is always satisfied.

For proof, see Theil (1971, p.396).

4. All of $W$, $LM$ and $LR$ are asymptotically distributed as $\chi^2(G)$ random variables under the null hypothesis $H_0 : h(\theta) = 0$ .

5. Under some comditions, we have $W \geq LR \geq LM$. See Engle (1981) "Wald, Likelihood and Lagrange Multiplier Tests in Econometrics," Chap. 13 in *Handbook of Econometrics*, Vol.2, Grilliches and Intriligator eds, North-Holland.

## 13.2 Example: W, LM and LR Tests

Date file $\Longrightarrow$ cons99.txt (same data as before)

Each column denotes year, nominal household expenditures (家計消費，10 billion yen), household disposable income (家計可処分所得，10 billion yen) and household expenditure deflator (家計消費デフレータ，1990=100) from the left.

| | | | |
|---|---|---|---|
| 1955 | 5430.1 | 6135.0 | 18.1 |
| 1956 | 5974.2 | 6828.4 | 18.3 |
| 1957 | 6686.3 | 7619.5 | 19.0 |
| 1958 | 7169.7 | 8153.3 | 19.1 |
| 1959 | 8019.3 | 9274.3 | 19.7 |
| 1960 | 9234.9 | 10776.5 | 20.5 |
| 1961 | 10836.2 | 12869.4 | 21.8 |
| 1962 | 12430.8 | 14701.4 | 23.2 |
| 1963 | 14506.6 | 17042.7 | 24.9 |
| 1964 | 16674.9 | 19709.9 | 26.0 |
| 1965 | 18820.5 | 22337.4 | 27.8 |
| 1966 | 21680.6 | 25514.5 | 29.0 |
| 1967 | 24914.0 | 29012.6 | 30.1 |
| 1968 | 28452.7 | 34233.6 | 31.6 |
| 1969 | 32705.2 | 39486.3 | 32.9 |
| 1970 | 37784.1 | 45913.2 | 35.2 |
| 1971 | 42571.6 | 51944.3 | 37.5 |
| 1972 | 49124.1 | 60245.4 | 39.7 |
| 1973 | 59366.1 | 74924.8 | 44.1 |
| 1974 | 71782.1 | 93833.2 | 53.3 |
| 1975 | 83591.1 | 108712.8 | 59.4 |
| 1976 | 94443.7 | 123540.9 | 65.2 |
| 1977 | 105397.8 | 135318.4 | 70.1 |
| 1978 | 115960.3 | 147244.2 | 73.5 |
| 1979 | 127600.9 | 157071.1 | 76.0 |
| 1980 | 138585.0 | 169931.5 | 81.6 |
| 1981 | 147103.4 | 181349.2 | 85.4 |
| 1982 | 157994.0 | 190611.5 | 87.7 |
| 1983 | 166631.6 | 199587.8 | 89.5 |
| 1984 | 175383.4 | 209451.9 | 91.8 |
| 1985 | 185335.1 | 220655.6 | 93.9 |
| 1986 | 193069.6 | 229938.8 | 94.8 |
| 1987 | 202072.8 | 235924.0 | 95.3 |
| 1988 | 212939.9 | 247159.7 | 95.8 |
| 1989 | 227122.2 | 263940.5 | 97.7 |
| 1990 | 243035.7 | 280133.0 | 100.0 |
| 1991 | 255531.8 | 297512.9 | 102.5 |
| 1992 | 265701.6 | 309256.6 | 104.5 |
| 1993 | 272075.3 | 317021.6 | 105.9 |
| 1994 | 279538.7 | 325655.7 | 106.7 |
| 1995 | 283245.4 | 331967.5 | 106.2 |
| 1996 | 291458.5 | 340619.1 | 106.0 |
| 1997 | 298475.2 | 345522.7 | 107.3 |

```
          PROGRAM
LINE  ********************************************
|       freq a;
|       smpl 1955 1997;
|       read(file='cons99.txt') year cons yd price;
|       rcons=cons/(price/100);
|       ryd=yd/(price/100);
|       lyd=log(ryd);
|       olsq rcons c ryd;
|       olsq @res @res(-1);
|       ar1  rcons c ryd;
|       olsq rcons c lyd;
|       param a1 0 a2 0 a3 1;
|       frml eq rcons=a1+a2*((ryd**a3)-1.)/a3;
|       lsq(tol=0.00001,maxit=100) eq;
|       a3=1.15;
|       rryd=((ryd**a3)-1.)/a3;
|       ar1  rcons c rryd;
|       end;
**************************************************
```

```
                              Equation   1
                              ============
                 Method of estimation = Ordinary Least Squares

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

       Mean of dep. var. = 146270.         LM het. test = .207443 [.649]
  Std. dev. of dep. var. = 79317.2         Durbin-Watson = .115101 [.000,.000]
Sum of squared residuals = .129697E+10  Jarque-Bera test = 9.47539 [.009]
   Variance of residuals = .316335E+08   Ramsey's RESET2 = 53.6424 [.000]
Std. error of regression = 5624.36       F (zero slopes) = 8311.90 [.000]
               R-squared = .995092        Schwarz B.I.C. = 435.051
      Adjusted R-squared = .994972        Log likelihood = -431.289

           Estimated    Standard
Variable  Coefficient     Error         t-statistic   P-value
C         -2919.54      1847.55         -1.58022      [.122]
RYD       .852879       .935486E-02     91.1696       [.000]
```

```
                                Equation   2
                                ============
                   Method of estimation = Ordinary Least Squares

Dependent variable: @RES
Current sample:  1956 to 1997
Number of observations:  42

       Mean of dep. var. = -95.5174
  Std. dev. of dep. var. = 5588.52
Sum of squared residuals = .146231E+09
   Variance of residuals = .356662E+07
Std. error of regression = 1888.55
               R-squared = .885884
      Adjusted R-squared = .885884
            LM het. test = .760256 [.383]
           Durbin-Watson = 1.40409 [.023,.023]
              Durbin's h = 1.97732 [.048]
         Durbin's h alt. = 1.91077 [.056]
        Jarque-Bera test = 6.49360 [.039]
         Ramsey's RESET2 = .186107 [.668]
          Schwarz B.I.C. = 377.788
          Log likelihood = -375.919

          Estimated    Standard
Variable  Coefficient    Error       t-statistic   P-value
@RES(-1)  .950693      .053301       17.8362       [.000]
```

```
                                   Equation   3
                                   ============
                 FIRST-ORDER SERIAL CORRELATION OF THE ERROR
                 Objective function:  Exact ML (keep first obs.)

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

       Mean of dep. var. = 146270.                R-squared = .999480
  Std. dev. of dep. var. = 79317.2       Adjusted R-squared = .999454
Sum of squared residuals = .145826E+09        Durbin-Watson = 1.38714
   Variance of residuals = .364564E+07      Schwarz B.I.C. = 391.061
Std. error of regression = 1909.36          Log likelihood = -385.419

                           Standard
Parameter  Estimate         Error       t-statistic   P-value
C          1672.42         6587.40      .253881       [.800]
RYD        .840011         .027182      30.9032       [.000]
RHO        .945025         .045843      20.6143       [.000]
```

```
                                        Equation   4
                                        ============
                          Method of estimation = Ordinary Least Squares

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

       Mean of dep. var. = 146270.          LM het. test = 2.21031 [.137]
  Std. dev. of dep. var. = 79317.2          Durbin-Watson = .029725 [.000,.000]
Sum of squared residuals = .256040E+11  Jarque-Bera test = 3.72023 [.156]
   Variance of residuals = .624487E+09   Ramsey's RESET2 = 344.855 [.000]
Std. error of regression = 24989.7       F (zero slopes) = 382.117 [.000]
               R-squared = .903100        Schwarz B.I.C. = 499.179
      Adjusted R-squared = .900737        Log likelihood = -495.418

           Estimated    Standard
Variable   Coefficient    Error        t-statistic   P-value
C          -.115228E+07  66538.5       -17.3175      [.000]
LYD        109305.       5591.69       19.5478       [.000]
```

```
                                NONLINEAR LEAST SQUARES
                                =======================
CONVERGENCE ACHIEVED AFTER  84 ITERATIONS

Number of observations = 43      Log likelihood = -414.362
        Schwarz B.I.C. = 420.004

                             Standard
Parameter  Estimate            Error          t-statistic   P-value
A1         16544.5           2615.60          6.32530       [.000]
A2         .063304           .024133          2.62307       [.009]
A3         1.21694           .031705          38.3839       [.000]

Standard Errors computed from quadratic form of analytic first derivatives
(Gauss)

Equation: EQ
Dependent variable: RCONS

       Mean of dep. var. = 146270.
  Std. dev. of dep. var. = 79317.2
Sum of squared residuals = .590213E+09
   Variance of residuals = .147553E+08
Std. error of regression = 3841.27
               R-squared = .997766
      Adjusted R-squared = .997655
            LM het. test = .174943 [.676]
           Durbin-Watson = .253234 [.000,.000]
```

```
                                   Equation   5
                                   ============
                  FIRST-ORDER SERIAL CORRELATION OF THE ERROR
                  Objective function:  Exact ML (keep first obs.)

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

       Mean of dep. var. = 146270.                 R-squared = .999470
  Std. dev. of dep. var. = 79317.2       Adjusted R-squared = .999443
Sum of squared residuals = .140391E+09         Durbin-Watson = 1.43657
   Variance of residuals = .350977E+07       Schwarz B.I.C. = 389.449
Std. error of regression = 1873.44           Log likelihood = -383.807

                            Standard
Parameter  Estimate          Error        t-statistic   P-value
C          12034.8        3346.47         3.59628       [.000]
RRYD       .140723        .282614E-02     49.7933       [.000]
RHO        .876924        .068199         12.8583       [.000]
```

1. `Equation 1` vs. `Equation 3` (Test of Serial Correlation)

   `Equation 1` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2 \texttt{RYD}_t + u_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

   `Equation 3` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2 \texttt{RYD}_t + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

   The null hypothesis is $H_0 : \ \rho = 0$

Restricted MLE $\Longrightarrow$ `Equation 1`

Unrestricted MLE $\Longrightarrow$ `Equation 3`

The log-likelihood function of `Equation 3` is:

$$\begin{aligned}\log L(\beta, \sigma_\epsilon^2, \rho) = &-\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\epsilon^2) + \frac{1}{2}\log(1-\rho^2) \\ &-\frac{1}{2\sigma_\epsilon^2}\sum_{t=1}^{n}(\mathtt{RCONS}_t^* - \beta_1\mathtt{CONST}_t^* - \beta_2\mathtt{RYD}_t^*)^2,\end{aligned}$$

where

$$\mathtt{RCONS}_t^* = \begin{cases}\sqrt{1-\rho^2}\,\mathtt{RCONS}_t, & \text{for } t = 1, \\ \mathtt{RCONS}_t - \rho\mathtt{RCONS}_{t-1}, & \text{for } t = 2, 3, \cdots, n,\end{cases}$$

$$\mathtt{CONST}_t^* = \begin{cases}\sqrt{1-\rho^2}, & \text{for } t = 1, \\ 1-\rho, & \text{for } t = 2, 3, \cdots, n,\end{cases}$$

$$\texttt{RYD}_t^* = \begin{cases} \sqrt{1-\rho^2}\texttt{RYD}_t, & \text{for } t = 1, \\ \texttt{RYD}_t - \rho\texttt{RYD}_{t-1}, & \text{for } t = 2, 3, \cdots, n. \end{cases}$$

- MLE with the restriction $\rho = 0$ (`Equation 1`) solves:

$$\max_{\beta, \sigma_\epsilon^2} \log L(\beta, \sigma_\epsilon^2, 0)$$

Restricted MLE $\Longrightarrow \tilde{\beta}, \tilde{\sigma}_\epsilon^2$

`Log of likelihood function = -431.289`

- MLE without the restriction $\rho = 0$ (`Equation 3`) solves:

$$\max_{\beta, \sigma_\epsilon^2, \rho} \log L(\beta, \sigma_\epsilon^2, \rho)$$

Unrestricted MLE $\Longrightarrow \hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho}$

`Log of likelihood function = -385.419`

The likelihood ratio test statistic is:

$$
\begin{aligned}
-2\log(\lambda) &= -2\log\Big(\frac{L(\tilde{\beta}, \tilde{\sigma}_\epsilon^2, 0)}{L(\hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho})}\Big) = -2\Big(\log L(\tilde{\beta}, \tilde{\sigma}_\epsilon^2, 0) - \log L(\hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho})\Big) \\
&= -2\Big(-431.289 - (-385.419)\Big) = 91.74.
\end{aligned}
$$

The asymptotic distribution is given by:

$$
-2\log(\lambda) \sim \chi^2(G),
$$

where $G$ is the number of the restrictions, i.e., $G = 1$ in this case.

The 1% upper probability point of $\chi^2(1)$ is 6.635.

$$
91.74 > 6.635
$$

Therefore, $H_0: \ \rho = 0$ is rejected.

There is serial correlation in the error term.

2. Equation 1 (Test of Serial Correlation $\longrightarrow$ Lagrange Multiplier Test)

Equation 2 is:

$$\texttt{@RES}_t = \rho\texttt{@RES}_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2),$$

where $\texttt{@RES}_t = \texttt{RCONS}_t - \hat{\beta}_1 - \hat{\beta}_2\texttt{RYD}_t$, and $\hat{\beta}_1$ and $\hat{\beta}_2$ are OLSEs.

The null hypothesis is $H_0:\ \rho = 0$

```
@RES(-1)   .950693        .053301        17.8362        [.000]
```

Therefore, the Lagrange multiplier test statistic is $17.8362^2 = 318.13 > 6.635$.

$H_0:\ \rho = 0$ is rejected.

3. Equation 3 (Test of Serial Correlation $\longrightarrow$ Wald Test)

`Equation 3` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2 \texttt{RYD}_t + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0 : \ \rho = 0$

```
RHO          .945025        .045843        20.6143         [.000]
```

The Wald test statistics is $20.6143^2 = 424.95$, which is compared with $\chi^2(1)$.

4. `Equation 1` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – linear):

   `NONLINEAR LEAST SQUARES` estimates:

$$\texttt{RCONS}_t = a1 + a2 \frac{\texttt{RYD}_t^{a3} - 1}{a3} + u_t.$$

When $a3 = 1$, we have:

$$\texttt{RCONS}_t = (a1 - a2) + a2\texttt{RYD}_t + u_t,$$

which is equivalent to `Equation 1`.

The null hypothesis is $H_0: \ a3 = 1$, where $G = 1$.

- MLE with $a3 = 1$ MLE (`Equation 1`)

  `Log of likelihood function = -431.289`

- MLE without $a3 = 1$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is given by:

$$-2\log(\lambda) = -2\Big(-431.289 - (-414.362)\Big) = 33.854.$$

The 1% upper probability point of $\chi^2(1)$ is 6.635.

$$33.854 > 6.635$$

$H_0: \ a3 = 1$ is rejected.

Therefore, the functional form of the regression model is not linear.

5. `Equation 4` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – log-linear):

   In `NONLINEAR LEAST SQUARES`, i.e.,

$$\mathtt{RCONS}_t = a1 + a2\frac{\mathtt{RYD}_t^{a3} - 1}{a3} + u_t,$$

   if $a3 = 0$, we have:

$$\mathtt{RCONS}_t = a1 + a2\log(\mathtt{RYD}_t) + u_t,$$

which is equivalent to `Equation 3`.

The null hypothesis is $H_0: \ a3 = 0$, where $G = 1$.

- MLE with $a3 = 0$ (`Equation 3`)

  `Log of likelihood function = -495.418`

- MLE without $a3 = 0$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\Big(-495.418 - (-414.362)\Big) = 162.112 > 6.635.$$

Therefore, $H_0: \ a3 = 0$ is rejected.

As a result, the functional form of the regression model is not log-linear, either.

6. `Equation 1` vs. `Equation 5` (Simultaneous Test of Serial Correlation and Linear Function):

`Equation 5` is:

$$\mathtt{RCONS}_t = a1 + a2\frac{\mathtt{RYD}_t^{a3} - 1}{a3} + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0:\ a3 = 1,\ \rho = 0$

Restricted MLE $\Longrightarrow$ `Equation 1`

Unrestricted MLE $\Longrightarrow$ `Equation 4`

**Remark:** In Lines 14–16 of `PROGRAM`, we have estimated `Equation 4`, given $a3 = 0.00, 0.01, 0.02, \cdots$.

As a result, $a3 = 1.15$ gives us the maximum log-likelihood.

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\big(-431.289 - (-383.807)\big) = 94.964.$$

$-2\log(\lambda) \sim \chi^2(2)$ in this case.

The 1% upper probability point of $\chi^2(2)$ is 9.210.

$$94.964 > 9.210$$

$H_0:\ a3 = 1, \rho = 0$ is rejected.

`Equation 3` vs. `Equation 5` vs. (Taking into account serially correlated errors, Choice of Functional Form – linear):

The null hypothesis is $H_0: \ a3 = 1, \ \rho = 0$

From `Equation 3`,

```
Log likelihood = -385.419
```

From `Equation 5`,

```
Log likelihood = -383.807
```

$$2(-383.807 - (-385.419)) = 3.224 < 6.635.$$

$H_0: \ a3 = 1$ is not rejected, given $\rho \neq 0$.

Thus, if serial correlation is taken into account, the regression model is linear.

1. **Asymptotic Normality of MLE:**

Let $\tilde{\theta}$ be MLE of $\theta$.

As $n$ goes to infinity, we have the following result:

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N\left(0, \lim_{n\to\infty}\left(\frac{I(\theta)}{n}\right)^{-1}\right),$$

where it is assumed that $\lim_{n\to\infty}\left(\frac{I(\theta)}{n}\right)$ converges.

That is, when $n$ is large, $\tilde{\theta}$ is approximately distributed as follows:

$$\tilde{\theta} \sim N\left(\theta, (I(\theta))^{-1}\right).$$

Suppose that $s(X) = \tilde{\theta}$.

When $n$ is large, $\mathrm{V}(s(X))$ is approximately equal to $(I(\theta))^{-1}$.

Practically, we utilize the following approximated distribution:

$$\tilde{\theta} \sim N\Big(\theta, (I(\tilde{\theta}))^{-1}\Big).$$

Then, we can obtain the significance test and the confidence interval for $\theta$

2. **Central Limit Theorem:** Let $X_1$, $X_2$, $\cdots$, $X_n$ be mutually independently distributed random variables with mean $\mathrm{E}(X_i) = \mu$ and variance $\mathrm{V}(X_i) = \sigma^2 < \infty$ for $i = 1, 2, \cdots, n$.

   Define $\overline{X} = (1/n) \sum_{i=1}^{n} X_i$.

   Then, the central limit theorem is given by:

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0, 1).$$

   Note that $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{V}(\overline{X}) = \sigma^2/n$.

That is,

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \sigma^2).$$

Note that $\mathrm{E}(\overline{X}) = \mu$ and $n\mathrm{V}(\overline{X}) = \sigma^2$.

In the case where $X_i$ is a vector of random variable with mean $\mu$ and variance $\Sigma < \infty$, the central limit theorem is given by:

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma).$$

Note that $\mathrm{E}(\overline{X}) = \mu$ and $n\mathrm{V}(\overline{X}) = \Sigma$.

3. **Central Limit Theorem II:** Let $X_1$, $X_2$, $\cdots$, $X_n$ be mutually independently distributed random variables with mean $\mathrm{E}(X_i) = \mu$ and variance $\mathrm{V}(X_i) = \sigma_i^2$ for $i = 1, 2, \cdots, n$.

   Assume:

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 < \infty.$$

   Define $\overline{X} = (1/n) \sum_{i=1}^{n} X_i$.

   The central limit theorem is given by:

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \sigma^2).$$

   Note that $\mathrm{E}(\overline{X}) = \mu$ and $n\mathrm{V}(\overline{X}) \longrightarrow \sigma^2$.

In the case where $X_i$ is a vector of random variable with mean $\mu$ and variance $\Sigma_i$, the central limit theorem is given by:

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma),$$

where $\Sigma = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Sigma_i < \infty$.

Note that $\mathrm{E}(\overline{X}) = \mu$ and $n\mathrm{V}(\overline{X}) \longrightarrow \Sigma$.

**[Review of Asymptotic Theories]**

- **Convergence in Probability** (確率収束) $X_n \longrightarrow a$, i.e., $X$ converges in probability to $a$, where $a$ is a fixed number.

- **Convergence in Distribution (分布収束)** $X_n \longrightarrow X$, i.e., $X$ converges in distribution to $X$. The distribution of $X_n$ converges to the distribution of $X$ as $n$ goes to infinity.

**Some Formulas**

$X_n$ and $Y_n$ : Convergence in Probability

$Z_n$ : Convergence in Distribution

- If $X_n \longrightarrow a$, then $f(X_n) \longrightarrow f(a)$.
- If $X_n \longrightarrow a$ and $Y_n \longrightarrow b$, then $f(X_n Y_n) \longrightarrow f(ab)$.
- If $X_n \longrightarrow a$ and $Z_n \longrightarrow Z$, then $X_n Z_n \longrightarrow aZ$, i.e., $aZ$ is distributed with mean $\mathrm{E}(aZ) = a\mathrm{E}(Z)$ and variance $\mathrm{V}(aZ) = a^2\mathrm{V}(Z)$.

**[End of Review]**

4. **Asymptotic Normality of MLE — Proof:**

The density (or probability) function of $X_i$ is given by $f(x_i; \theta)$.

The likelihood function is: $L(\theta; x) \equiv f(x; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$,

where $x = (x_1, x_2, \cdots, x_n)$.

MLE of $\theta$ results in the following maximization problem:

$$\max_{\theta} \quad \log L(\theta; x).$$

A solution of the above problem is given by MLE of $\theta$, denoted by $\tilde{\theta}$.

That is, $\tilde{\theta}$ is given by the $\theta$ which satisfies the following equation:

$$\frac{\partial \log L(\theta; x)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \log f(x_i; \theta)}{\partial \theta} = 0.$$

Replacing $x_i$ by the underlying random variable $X_i$, $\dfrac{\partial \log f(X_i;\theta)}{\partial\theta}$ is taken as the $i$th random variable, i.e., $X_i$ in the **Central Limit Theorem II**.

Consider applying **Central Limit Theorem II**.

In this case, we need the following expectation and variance:

$$\mathrm{E}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big) \qquad \text{and} \qquad \mathrm{V}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big).$$

Defining the variance:

$$\mathrm{V}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big) = \Sigma_i,$$

we can rewrite the information matrix as follows:

$$\begin{aligned} I(\theta) &= \mathrm{V}\Big(\frac{\partial \log L(\theta;X)}{\partial\theta}\Big) = \mathrm{V}\Big(\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big) \\ &= \sum_{i=1}^{n}\mathrm{V}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big) = \sum_{i=1}^{n}\Sigma_i \end{aligned}$$

The third equality holds when $X_1, X_2, \cdots, X_n$ are mutually independent.

Note that $\mathrm{E}\Big(\dfrac{\partial \log L(\theta; X)}{\partial \theta}\Big) = 0$ and $\mathrm{V}\Big(\dfrac{\partial \log L(\theta; X)}{\partial \theta}\Big) = I(\theta)$.

$$\frac{1}{n}\frac{\partial \log L(\theta; X)}{\partial \theta} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i; \theta)}{\partial \theta}$$

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i; \theta)}{\partial \theta} - \mathrm{E}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i; \theta)}{\partial \theta}\Big)\right) \longrightarrow N(0, \Sigma),$$

where

$$n\mathrm{V}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i; \theta)}{\partial \theta}\Big) = \frac{1}{n}\mathrm{V}\Big(\sum_{i=1}^{n}\frac{\partial \log f(X_i; \theta)}{\partial \theta}\Big) = \frac{1}{n}\mathrm{V}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big)$$

$$= \frac{1}{n}I(\theta) \longrightarrow \Sigma.$$

That is,

$$\frac{1}{\sqrt{n}} \frac{\partial \log L(\theta; X)}{\partial \theta} \longrightarrow N(0, \Sigma),$$

where $X = (X_1, X_2, \cdots, X_n)$.

Now, consider replacing $\theta$ by $\tilde{\theta}$, i.e.,

$$\frac{1}{\sqrt{n}} \frac{\partial \log L(\tilde{\theta}; X)}{\partial \theta},$$

which is expanded around $\tilde{\theta} = \theta$ as follows:

$$0 = \frac{1}{\sqrt{n}} \frac{\partial \log L(\tilde{\theta}; X)}{\partial \theta} \approx \frac{1}{\sqrt{n}} \frac{\partial \log L(\theta; X)}{\partial \theta} + \frac{1}{\sqrt{n}} \frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'} (\tilde{\theta} - \theta).$$

Therefore,

$$\frac{1}{\sqrt{n}} \frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'} (\tilde{\theta} - \theta) \approx -\frac{1}{\sqrt{n}} \frac{\partial \log L(\theta; X)}{\partial \theta} \longrightarrow N(0, \Sigma).$$

The left-hand side is rewritten as:

$$\frac{1}{\sqrt{n}}\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'} = \sqrt{n}\frac{1}{n}\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}(\tilde{\theta} - \theta).$$

Then,

$$\begin{aligned}\sqrt{n}(\tilde{\theta} - \theta) &\approx -\Big(\frac{1}{n}\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\Big)^{-1}\Big(\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta; X)}{\partial\theta}\Big) \\ &\longrightarrow N(0, \Sigma^{-1}\Sigma\Sigma^{-1}) = N(0, \Sigma^{-1}).\end{aligned}$$

Note that

$$\frac{1}{n}\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'} \longrightarrow \lim_{n\to\infty}\frac{1}{n}\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\Big) = \Sigma,$$

and $\Big(\frac{1}{n}\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\Big)^{-1}\Big(\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta; X)}{\partial\theta}\Big)$ has the same asymptotic distribution as $\Sigma^{-1}\Big(\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta; X)}{\partial\theta}\Big)$.

# Exam — Aug. 4, 2016 (AM8:50-10:20)

- 60 - 70% from two homeworks including optional an additional questions (2 つの宿題から 60 - 70%)
- 30 - 40% of new questions (30 - 40% の新しい問題)
- Questions are written in English, and answers should be in English or Japanese. (出題は英語，解答は英語または日本語)
- With no carrying in (持ち込みなし)