

# Homework (Due: July 7, 2016 AM10:20)

Let  $y$ ,  $X$ ,  $\beta$  and  $u$  be a  $n \times 1$  vector, a  $n \times k$  matrix, a  $k \times 1$  vector and a  $n \times 1$  vector, respectively, where  $X$  is nonstochastic,  $u$  is an error term, and  $\beta$  is a parameter.

Consider the regression model:  $y = X\beta + u$ .

- $E(u) = 0$  and  $V(u) = \sigma^2 I_n$  are assumed.
  - (1) Derive an OLS estimator of  $\beta$ , which is denoted by  $\hat{\beta}$ .
  - (2) Obtain the mean and variance of  $\hat{\beta}$ .
  - (3) Prove that  $\hat{\beta}$  is a best linear unbiased estimator.
  - (4) Show that  $\hat{\beta}$  is a consistent estimator of  $\beta$ . Which assumptions are used?
  - (5) As  $n$  goes to infinity, what is a distribution of  $\sqrt{n}(\hat{\beta} - \beta)$ ?
  - (6) Let  $R$  and  $r$  be a  $G \times k$  matrix and a  $G \times 1$  vector, where  $G \leq k$  and  $\text{Rank}(R) = G$ . Suppose that the linear restriction is given by  $R\beta = r$  ( $G$  restrictions). Derive the restricted OLS estimator, denoted by  $\tilde{\beta}$ .
  - (7) Consider  $\hat{\sigma}^2 = \frac{1}{n-k} \hat{u}'\hat{u}$  as an estimator of  $\sigma^2$ . Prove that  $\hat{\sigma}^2$  is an unbiased estimator of  $\sigma^2$ .
- $u \sim N(0, \sigma^2 I_n)$  is assumed.
  - (8) Derive distributions of  $\hat{\beta}$  and  $\frac{(n-k)\hat{\sigma}^2}{\sigma^2}$ .
  - (9) Show that  $\hat{\sigma}^2$  is a consistent estimator of  $\sigma^2$ . Assume that  $u \sim N(0, \sigma^2 I_n)$ .
  - (10) Using the distributions of  $\hat{\beta}$  and  $\frac{(n-k)\hat{\sigma}^2}{\sigma^2}$  obtained in Question (8), explain how to test  $H_0 : R\beta = r$  against  $H_1 : R\beta \neq r$ .
  - (11) Using  $\hat{u} = y - X\hat{\beta}$  and  $\tilde{u} = y - X\tilde{\beta}$ , show that the test statistic obtained in Question (10) is given by  $\frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(n-k)}$ .
  - (12) Obtain ML estimators of  $\beta$  and  $\sigma^2$ , which are denoted by  $\bar{\beta}$  and  $\bar{\sigma}^2$ .
  - (13) Derive a distribution of  $\sqrt{n} \begin{pmatrix} \bar{\beta} - \beta \\ \bar{\sigma}^2 - \sigma^2 \end{pmatrix}$  when  $n$  goes to infinity.

- $u_t = \rho u_{t-1} + \epsilon_t$ ,  $t = 1, 2, \dots, n$ , and  $\epsilon \sim N(0, \sigma^2 I_n)$  are assumed.

(14) Obtain the unconditional distribution of  $u_t$ .

(15) Obtain the conditional distribution of  $u_t$ , given  $u_{t-1}, u_{t-2}, \dots, u_1$ .

(16) Derive the likelihood function of  $\beta$ ,  $\sigma^2$  and  $\rho$ , given data  $y$  and  $X$ .

(17) Explain how to obtain the maximum likelihood estimators of  $\beta$ ,  $\sigma^2$  and  $\rho$ .