8.3 GLS — Review

Regression model:

$$y = X\beta + u, \qquad\qquad u \sim N(0, \Omega),$$

where y, X, β, u, 0 and Ω are $n \times 1$, $n \times k$, $k \times 1$, $n \times 1$, $n \times 1$, and $n \times n$, respectively.

We solve the following minimization problem:

$$\min_{\beta} (y - X\beta)'\Omega^{-1}(y - X\beta).$$

Let b be a solution of the above minimization problem.

GLS estimator of β is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

In general, when Ω is symmetric, Ω is decomposed as follows.

$$\Omega = A'\Lambda A$$

Λ is a diagonal matrix, where the diagonal elements of Λ are given by the eigen values.

A is a matrix consisting of eigen vectors.

When Ω is a positive definite matrix, all the diagonal elements of Λ are positive.

There exists P such that $\Omega = PP'$ (i.e., take $P = A'\Lambda^{1/2}$). $\implies P^{-1}\Omega P'^{-1} = I_n$

Multiply P^{-1} on both sides of $y = X\beta + u$.

We have:

$$y^\star = X^\star\beta + u^\star,$$

where $\quad y^\star = P^{-1}y, \quad X^\star = P^{-1}X, \quad$ and $\quad u^\star = P^{-1}u$.

The variance of $u^\star$ is:

$$\mathrm{V}(u^\star) = \mathrm{V}(P^{-1}u) = P^{-1}\mathrm{V}(u)P'^{-1} = \sigma^2 P^{-1}\Omega P'^{-1} = \sigma^2 I_n.$$

because $\Omega = PP'$, i.e., $P^{-1}\Omega P'^{-1} = I_n$.

Accordingly, the regression model is rewritten as:

$$y^\star = X^\star\beta + u^\star, \qquad u^\star \sim (0, \sigma^2 I_n)$$

Apply OLS to the above model.

Let b be as estimator of β from the above model.

That is, the minimization problem is given by:

$$\min_b \; (y^\star - X^\star b)'(y^\star - X^\star b),$$

which is equivalent to:

$$\min_b \; (y - Xb)'\Omega^{-1}(y - Xb).$$

Solving the minimization problem above, we have the following estimator:

$$
\begin{aligned}
b &= (X^{\star\prime}X^{\star})^{-1}X^{\star\prime}y^{\star} \\
&= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y,
\end{aligned}
$$

which is called GLS (Generalized Least Squares) estimator.

b is rewritten as follows:

$$
b = \beta + (X^{\star\prime}X^{\star})^{-1}X^{\star\prime}u^{\star} = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u
$$

The mean and variance of b are given by:

$$
\begin{aligned}
&\mathrm{E}(b) = \beta, \\
&\mathrm{V}(b) = \sigma^2(X^{\star\prime}X^{\star})^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}.
\end{aligned}
$$

Suppose that the regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

In this case, when we use OLS, what happens?

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$$

$$\mathrm{V}(\hat{\beta}) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$

● Compare GLS and OLS.

Expectation:

$$\mathrm{E}(\hat{\beta}) = \beta, \qquad \text{and} \qquad \mathrm{E}(b) = \beta$$

Thus, both $\hat{\beta}$ and b are unbiased estimator.

Variance:

$$\mathrm{V}(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}$$

$$\mathrm{V}(b) = \sigma^2 (X'\Omega^{-1} X)^{-1}$$

Which is more efficient, OLS or GLS?.

$$\begin{aligned}
\mathrm{V}(\hat{\beta}) - \mathrm{V}(b) &= \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1} - \sigma^2 (X'\Omega^{-1} X)^{-1} \\
&= \sigma^2 \Big((X'X)^{-1} X' - (X'\Omega^{-1} X)^{-1} X'\Omega^{-1} \Big) \Omega \\
&\qquad \times \Big((X'X)^{-1} X' - (X'\Omega^{-1} X)^{-1} X'\Omega^{-1} \Big)' \\
&= \sigma^2 A \Omega A'
\end{aligned}$$

Ω is the variance-covariance matrix of u, which is a positive definite matrix.

Therefore, except for $\Omega = I_n$, $A\Omega A'$ is also a positive definite matrix.

This implies that $\mathrm{V}(\hat{\beta}_i) - \mathrm{V}(b_i) > 0$ for the ith element of β.

Accordingly, b is more efficient than $\hat{\beta}$.

If $u \sim N(0, \sigma^2\Omega)$, then $b \sim N(\beta, \sigma^2(X'\Omega^{-1}X)^{-1})$.

● Maximum Likelihood Estimation (MLE):

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

$\mathrm{E}(y) = X\beta$ and $\mathrm{V}(y) = \sigma^2\Omega$

$$f(y) = (2\pi\sigma^2)^{-n/2}|\Omega|^{-1/2}\exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)\Omega^{-1}(y - X\beta)'\Big) = L(\beta, \sigma^2)$$

b is equivalent to MLE.

8.4 Panel Model Basic

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where i indicates individual and t denotes time.

There are n observations for each t.

u_{it} indicates the error term, assuming that $\mathrm{E}(u_{it}) = 0$, $\mathrm{V}(u_{it}) = \sigma_u^2$ and $\mathrm{Cov}(u_{it}, u_{js}) = 0$ for $i \neq j$ and $t \neq s$.

v_i denotes the individual effect, which is fixed or random.

8.4.1 Fixed Effect Model (固定効果モデル)

In the case where v_i is fixed, the case of $v_i = z_i\alpha$ is included.

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

$$\overline{y}_i = \overline{X}_i\beta + v_i + \overline{u}_i, \qquad i = 1, 2, \cdots, n,$$

where $\overline{y}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} y_{it}$, $\overline{X}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} X_{it}$, and $\overline{u}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} u_{it}$.

$$(y_{it} - \overline{y}_i) = (X_{it} - \overline{X}_i)\beta + (u_{it} - \overline{u}_i), \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

Taking an example of y, the left-hand side of the above equation is rewritten as:

$$y_{it} - \overline{y}_i = y_{it} - \frac{1}{T} 1_T' y_i,$$

where $1_T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, which is a $T \times 1$ vector, and $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$.

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = I_T y_i - 1_T \overline{y}_i = I_T y_i - \frac{1}{T} 1_T 1_T' y_i = (I_T - \frac{1}{T} 1_T 1_T') y_i$$

Thus,

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = \begin{pmatrix} X_{i1} - \overline{X}_i \\ X_{i2} - \overline{X}_i \\ \vdots \\ X_{iT} - \overline{X}_i \end{pmatrix} \beta + \begin{pmatrix} u_{i1} - \overline{u}_i \\ u_{i2} - \overline{u}_i \\ \vdots \\ u_{iT} - \overline{u}_i \end{pmatrix}, \qquad i = 1, 2, \cdots, n,$$

which is re-written as:

$$(I_T - \frac{1}{T}1_T 1_T')y_i = (I_T - \frac{1}{T}1_T 1_T')X_i\beta + (I_T - \frac{1}{T}1_T 1_T')u_i, \qquad i = 1, 2, \cdots, n,$$

i.e.,

$$D_T y_i = D_T X_i \beta + D_T u_i, \qquad i = 1, 2, \cdots, n,$$

where $D_T = (I_T - \frac{1}{T}1_T 1_T')$, which is a $T \times T$ matrix.

Note that $D_T D_T' = D_T$, i.e., D_T is a symmetric and idempotent matrix.

Using the matrix form for $i = 1, 2, \cdots, n$, we have:

$$\begin{pmatrix} D_T y_1 \\ D_T y_2 \\ \vdots \\ D_T y_n \end{pmatrix} = \begin{pmatrix} D_T X_1 \\ D_T X_2 \\ \vdots \\ D_T X_n \end{pmatrix} \beta + \begin{pmatrix} D_T u_1 \\ D_T u_2 \\ \vdots \\ D_T u_n \end{pmatrix},$$