8.5 Consistency and Asymptotic Normality of OLSE — Review

Regression model: $y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n)$.

Consistency:

1. Let $\hat{\beta}_n = (X'X)^{-1}X'y$ be the OLS with sample size n.

   Consistency: As n is large, $\hat{\beta}_n$ converges to β.

2. Assume the stationarity assumption for X, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

   Then, we have the following result:

$$\frac{1}{n}X'u \longrightarrow 0.$$

Proof:

According to Chebyshev's inequality, for $g(Z) \geq 0$,

$$P(g(Z) \geq k) \leq \frac{\mathrm{E}(g(Z))}{k},$$

where k is a positive constant.

Set $g(Z) = Z'Z$, and $Z = \dfrac{1}{n}X'u$.

Apply Chebyshev's inequality.

$$\begin{aligned}\mathrm{E}\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u\Big) &= \frac{1}{n^2}\mathrm{E}\Big(u'XX'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(u'XX'u)\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(XX'uu')\Big)\\ &= \frac{1}{n^2}\mathrm{tr}\Big(XX'\mathrm{E}(uu')\Big) = \frac{\sigma^2}{n^2}\mathrm{tr}(XX') = \frac{\sigma^2}{n^2}\mathrm{tr}(X'X) = \frac{\sigma^2}{n}\mathrm{tr}(\frac{1}{n}X'X).\end{aligned}$$

Therefore,

$$P\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u \geq k\Big) \leq \frac{\sigma^2}{nk}\mathrm{tr}(\frac{1}{n}X'X) \longrightarrow 0 \times \mathrm{tr}(M_{xx}) = 0.$$

Note that from the assumption,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Therefore, we have:

$$(\frac{1}{n}X'u)'\frac{1}{n}X'u \longrightarrow 0,$$

which implies:

$$\frac{1}{n}X'u \longrightarrow 0,$$

because $(\frac{1}{n}X'u)'\frac{1}{n}X'u$ indicates a quadratic form.

3. Note that $\frac{1}{n}X'X \longrightarrow M_{xx}$ results in $(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$.

$\Longrightarrow$ Slutsky's Theorem

(*) **Slutsky's Theorem** $g(\hat{\theta}) \longrightarrow g(\theta)$, when $\hat{\theta} \longrightarrow \theta$.

4. OLS is given by:

$$\hat{\beta}_n = \beta + (X'X)^{-1}X'u = \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat{\beta}_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consitent estimator.

Asymptotic Normality:

1. Asymptotic Normality of OLSE

$$\sqrt{n}(\hat{\beta}_n - \beta) \longrightarrow N(0,\ \sigma^2 M_{xx}^{-1}), \quad \text{when } n \longrightarrow \infty.$$

2. **Central Limit Theorem:** Greenberg and Webster (1983)

   $Z_1, Z_2, \cdots, Z_n$ are mutually indelendently distributed with mean μ and variance Σ_i.

   Then, we have the following result:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i - \mu) \longrightarrow N(0, \Sigma),$$

   where

$$\Sigma = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \Sigma_i \right).$$

   The distribution of Z_i is not assumed.

3. Define $Z_i = x_i' u_i$. Then, $\Sigma_i = \mathrm{Var}(Z_i) = \sigma^2 x_i' x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$
\begin{aligned}
\mathrm{Var}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\Big) &= \mathrm{E}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\big)'\Big) \\
&= \big(\frac{1}{n}X'X\big)^{-1}\big(\frac{1}{n}X'\mathrm{E}(uu')X\big)\big(\frac{1}{n}X'X\big)^{-1} \\
&= \sigma^2\big(\frac{1}{n}X'X\big)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}.
\end{aligned}
$$

Therefore,

$$
\sqrt{n}(\hat{\beta}-\beta) \longrightarrow N(0, \sigma^2 M_{xx}^{-1})
$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

8.6 Instrumental Variable (IV) Method (操作変数法 or IV 法) — Review

Instrumental Variable (IV)

1. Consider the regression model: $y = X\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE of β is inconsistent.

2. **Proof:**

$$\hat{\beta} = \beta + (\frac{1}{n}X'X)^{-1}\frac{1}{n}X'u \;\longrightarrow\; \beta + M_{xx}^{-1}M_{xu},$$

   where

$$\frac{1}{n}X'X \;\longrightarrow\; M_{xx}, \qquad \frac{1}{n}X'u \;\longrightarrow\; M_{xu} \neq 0$$

3. Find the Z which satisfies $\frac{1}{n}Z'u \;\longrightarrow\; M_{zu} = 0.$

Multiplying Z' on both sides of the regression model: $y = X\beta + u$,

$$Z'y = Z'X\beta + Z'u$$

Dividing n on both sides of the above equation, we take plim on both sides.

Then, we obtain the following:

$$\text{plim}\left(\frac{1}{n}Z'y\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta + \text{plim}\left(\frac{1}{n}Z'u\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta.$$

Accordingly, we obtain:

$$\beta = \left(\text{plim}\left(\frac{1}{n}Z'X\right)\right)^{-1} \text{plim}\left(\frac{1}{n}Z'y\right).$$

Therefore, we consider the following estimator:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

which is taken as an estimator of β.

$\Longrightarrow$ **Instrumental Variable Method (操作変数法 or IV 法)**

4. Assume the followings:

$$\frac{1}{n}Z'X \longrightarrow M_{zx}, \qquad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \qquad \frac{1}{n}Z'u \longrightarrow 0$$

5. **Asymptotic Distribution of β_{IV}:**

$$\beta_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u,$$

which is rewritten as:

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big)$$

Applying the Central Limit Theorem to $\Big(\frac{1}{\sqrt{n}}Z'u\Big)$, we have the following result:

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Therefore,

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}'^{-1})$$

$\Longrightarrow$ Consistency and Asymptotic Normality

6. The variance of β_{IV} is given by:

$$\mathrm{V}(\beta_{IV}) = s^2 (Z'X)^{-1} Z'Z (X'Z)^{-1},$$

where

$$s^2 = \frac{(y - X\beta_{IV})'(y - X\beta_{IV})}{n-k}.$$

8.7 Two-Stage Least Squares Method (2段階最小二乗法, 2SLS or TSLS) — Review

1. Regression Model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I),$$

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE is not consistent.

2. Find the variable Z which satisfies $\dfrac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

3. Use $Z = \hat{X}$ for the instrumental variable.

   $\hat{X}$ is the predicted value which regresses X on the other exogenous variables, say W.

   That is, consider the following regression model:

$$X = WB + V.$$

Estimate B by OLS.

Then, we obtain the prediction:

$$\hat{X} = W\hat{B},$$

where $\hat{B} = (W'W)^{-1}W'X$.

Or, equivalently,

$$\hat{X} = W(W'W)^{-1}W'X.$$

$\hat{X}$ is used for the instrumental variable of X.

4. The IV method is rewritten as:

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'y.$$

Furthermore, β_{IV} is written as follows:

$$\beta_{IV} = \beta + (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'u.$$

Therefore, we obtain the following expression:

$$\sqrt{n}(\beta_{IV} - \beta) = \left(\left(\frac{1}{n}X'W\right)\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{1}{n}XW'\right)'\right)^{-1}\left(\frac{1}{n}X'W\right)\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{1}{\sqrt{n}}W'u\right)$$
$$\longrightarrow N\left(0, \sigma^2(M_{xw}M_{ww}^{-1}M'_{xw})^{-1}\right).$$

5. Clearly, there is no correlation between W and u at least in the limit, i.e.,

$$\text{plim}\left(\frac{1}{n}W'u\right) = 0.$$

6. **Remark:**

$$\hat{X}'X = X'W(W'W)^{-1}W'X = X'W(W'W)^{-1}W'W(W'W)^{-1}W'X = \hat{X}'\hat{X}.$$

Therefore,

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (\hat{X}'\hat{X})^{-1}\hat{X}'y,$$

which implies the OLS estimator of β in the regression model: $y = \hat{X}\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

Example:

$$y_t = \alpha x_t + \beta z_t + u_t, \qquad u_t \sim (0, \sigma^2).$$

Suppose that x_t is correlated with u_t but z_t is not correlated with u_t.

- 1st Step:

Estimate the following regression model:

$$x_t = \gamma w_t + \delta z_t + \cdots + v_t,$$

by OLS. $\Longrightarrow$ Obtain $\hat{x}_t$ through OLS.

- 2nd Step:

Estimate the following regression model:

$$y_t = \alpha \hat{x}_t + \beta z_t + u_t,$$

by OLS. $\Longrightarrow$ α_{iv} and β_{iv}

Note as follows. Estimate the following regression model:

$$z_t = \gamma_2 w_t + \delta_2 z_t + \cdots + v_{2t},$$

by OLS.

$\Longrightarrow$ $\hat{\gamma}_2 = 0$, $\hat{\delta}_2 = 1$, and the other coefficient estimates are zeros. i.e., $\hat{z}_t = z_t$.

Eviews Command:

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tsls y x z @ w z ...
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