7 Time Series Analysis (時系列分析)

7.1 Introduction

1. **Stationarity (定常性) :**

   Let $y_1, y_2, \cdots, y_T$ be time series data.

   (a) **Weak Stationarity (弱定常性) :**

   $$\mathrm{E}(y_t) = \mu,$$

   $$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

   The first moment does not depend on time.

   The second moment depends only on time difference.

(b) **Strong Stationarity (強定常性) :**

Let $f(y_{t_1}, y_{t_2}, \cdots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \cdots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \cdots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \cdots, y_{t_r+\tau})$$

All the moments are same for all τ.

2. **Ergodicity (エルゴード性) :**

As time difference between two data is large, the two data become independent.

$y_1, y_2, \cdots, y_T$ is said to be ergodic in mean when $\overline{y}$ converges in probability to $\mathrm{E}(y_t)$.

3. **Auto-covariance Function (自己共分散関数) :**

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

$$\gamma(\tau) = \gamma(-\tau)$$

4. **Auto-correlation Function (自己相関関数) :**

$$\rho(\tau) = \frac{\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\mathrm{Var}(y_t)}\sqrt{\mathrm{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-\tau}) = \gamma(0)$.

5. **Sample Mean (標本平均) :**

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

6. **Sample Auto-covariance (標本自己共分散) :**

$$\hat{\gamma}(\tau) = \frac{1}{T}\sum_{t=\tau+1}^{T} (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

7. **Correlogram (コレログラム, or 標本自己相関関数) :**

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

8. **Lag Operator (ラグ作要素) :**

$$L^{\tau} y_t = y_{t-\tau}, \qquad \tau = 1, 2, \cdots$$

9. **Likelihood Function (尤度関数)** — Innovation Form :

The joint distribution of $y_1, y_2, \cdots, y_T$ is written as:

$$\begin{aligned}
f(y_1, , y_2, \cdots, y_T) &= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1}, \cdots, y_1) \\
&= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1} | y_{T-2}, \cdots, y_1) f(y_{T-2}, \cdots, y_1) \\
&\;\vdots \\
&= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1} | y_{T-2}, \cdots, y_1) \ \cdots \ f(y_2 | y_1) f(y_1) \\
&= f(y_1) \prod_{t=2}^{T} f(y_t | y_{t-1}, \cdots, y_1).
\end{aligned}$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \cdots, y_T) = \log f(y_1) + \sum_{t=2}^{T} \log f(y_t | y_{t-1}, \cdots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \cdots, y_1)$ is given by the normal distribution with conditional mean $\mathrm{E}(y_t|y_{t-1}, \cdots, y_1)$ and conditional variance $\mathrm{Var}(y_t|y_{t-1}, \cdot$

7.2 Time Series Models (時系列モデル)

Autoregressive Model (自己回帰モデル or AR モデル): AR(p)

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$

Moving Average Model (移動平均モデル or MA モデル): MA(q)

$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$

ARMA Model: ARMA(p, q)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

ARIMA Model: ARIMA(p, d, q)

$$\begin{aligned}
\Delta y_t &= y_t - y_{t-1} = (1 - L) y_t, \\
\Delta^2 y_t &= \Delta y_t - \Delta y_{t-1} = (1 - L)^2 y_t, \\
&\vdots \\
\Delta^d y_t &= (1 - L)^d y_t.
\end{aligned}$$

$$\Delta^d y_t \sim \text{ARMA}(p, q) \iff y_t \sim \text{ARIMA}(p, d, q)$$

$$\Delta^d y_t = \phi_1 \Delta^d y_{t-1} + \phi_2 \Delta^d y_{t-2} + \cdots + \phi_p \Delta^d y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

SARIMA Model: SARIMA(p, d, q)

${}^{s}\Delta y_t = y_t - y_{t-s}$, $s = 4$ for quarterly data $s = 12$ for monthly data

$$ {}^{s}\Delta\Delta^{d} y_t \;\sim\; \text{ARMA}(p, q) \iff y_t \;\sim\; \text{SARIMA}(p, d, q) $$

$$ {}^{s}\Delta\Delta^{d} y_t = \phi_1 {}^{s}\Delta\Delta^{d} y_{t-1} + \phi_2 {}^{s}\Delta\Delta^{d} y_{t-2} + \cdots + \phi_p {}^{s}\Delta\Delta^{d} y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} $$

7.3 Autoregressive Model (自己回帰モデル or AR モデル)

1. **AR(p) Model :**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. **Stationarity (定常性) :**

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one in absolute value, y_t is stationary.

Suppose that the p solutions include imaginary numbers.

When the p solutions are outside unit circle, y_t is stationary.

3. **Partial Autocorrelation Coefficient (偏自己相関係数), $\boldsymbol{\phi_{k,k}}$:**

The partial autocorrelation coefficient between y_t and y_{t-k}, denoted by $\phi_{k,k}$, is a measure of strength of the relationship between y_t and y_{t-k}, after removing influence of $y_{t-1}, \cdots, y_{t-k+1}$.

$$\phi_{1,1} = \rho(1)$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix}$$

Use Cramer's rule (クラメールの公式) to obtain $\phi_{k,k}$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

2. Rewriting the AR(1) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\
&\ \vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.
\end{aligned}
$$

As s is large, ϕ_1^s approaches zero. $\Longrightarrow$ Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$
y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots
$$

MA representation of AR model.

(MA will be discussed later.)

4. Mean of AR(1) process, μ

$$\mu = \mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots)$$
$$= \mathrm{E}(\epsilon_t) + \phi_1 \mathrm{E}(\epsilon_{t-1}) + \phi_1^2 \mathrm{E}(\epsilon_{t-2}) + \cdots = 0$$

5. Autocovariance and autocorrelation functions of the AR(1) process:

   Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

   Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}
\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\
&= \mathrm{E}\Big((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}) y_{t-\tau}\Big) \\
&= \phi_1^\tau \mathrm{E}(y_{t-\tau} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) + \phi_1 \mathrm{E}(\epsilon_{t-1} y_{t-\tau}) + \cdots + \phi_1^{\tau-1} \mathrm{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\
&= \phi_1^\tau \gamma(0).
\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^\tau.$$

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$\mathrm{E}(y_t y_{t-\tau}) = \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2 = \phi_1^2 \gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1 \gamma(0)$.

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2}$$

6. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2)-\rho(1)^2}{1-\rho(1)^2} = 0$$

7. Estimation of AR(1) model:

   (a) Likelihood function

$$\log f(y_T, \cdots, y_1) = \log f(y_1) + \sum_{t=1}^{T} \log f(y_t | y_{t-1}, \cdots, y_1)$$