8 Vector Autoregressive (VAR) Model – Causality, Impulse Response Function and etc

Vector Autoregressive Process:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where

$$y_t : k \times 1, \qquad \mu : k \times 1, \qquad \epsilon_t : k \times 1, \qquad \phi_i : k \times k.$$

Rewriting the above equation,

$$\phi(L) y_t = \mu + \epsilon_t,$$

where $\phi(L) = I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$.

VAR(1) Model:

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \qquad \text{i.e.,} \qquad (I_k - \phi_1 L) y_t = \epsilon_t.$$

When y_t is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L)^{-1} \epsilon_t \\ &= (I_k + \phi_1 L + \phi_1^2 L^2 + \phi_1^3 L^3 + \ \cdots) \epsilon_t \\ &= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \ \cdots \end{aligned}$$

VAR(1)=VMA(∞)

VAR(2) Model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \qquad \text{i.e.,} \qquad (I_k - \phi_1 L - \phi_2 L^2) y_{t-1} = \epsilon_t.$$

When y_t is stationary, we obtain:

$$y_{t-1} = (I_k - \phi_1 L - \phi_2 L^2)^{-1} \epsilon_t$$
$$= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots$$

VAR(2)=VMA(∞)

VAR(p) Model:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

i.e.,

$$(I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) y_{t-1} = \epsilon_t.$$

When y_t is stationary, we obtain:

$$y_t = (I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1} \epsilon_t$$
$$= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots$$

VAR(p)=VMA(∞)

8.1 Autocovariance Matrix and Autocorrelation Matrix

Let y_t be a $k \times 1$ vector.

Autocovariance Function Matrix:

$$\Gamma(\tau) = \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)'), \qquad \tau = 0, 1, 2, \cdots,$$

where $\mathrm{E}(y_t) = \mu$. $\Gamma(\tau)$ is a $k \times k$ matrix.

$$\Gamma(\tau) = \Gamma(-\tau)'$$

Autocorrelation Function Matrix:

$$\rho(\tau) = D^{-1/2}\Gamma(\tau)D^{-1/2},$$

where the (i, j)th element of D is given by $\gamma_{ii}(\tau) = \mathrm{V}(y_{it})$ for $i = j$ and zero otherwise.

$$\rho(\tau) = \rho(-\tau)'$$

8.2 Granger Cuasality Test (グレンジャー因果性テスト)

Consider a bivariate case.

Unrestricted Model (Sum of Squared Residuals, denoted by SSR_1):

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & \phi_{12,1} \\ \phi_{21,1} & \phi_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,p} & \phi_{12,p} \\ \phi_{21,p} & \phi_{22,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$H_0: \ \phi_{12,1} = \phi_{12,2} = \cdots = \phi_{12,p} = 0$

When H_0 is correct, we say there is no causality from y_2 to y_1.

$\Longrightarrow$ Granger Causality Test.

Restricted Model (Sum of Squared Residuals, denoted by SSR_0):

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & 0 \\ \phi_{21,1} & \phi_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,p} & 0 \\ \phi_{21,p} & \phi_{22,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Asymptotically, we have the following distribution:

$$F = \frac{(\mathrm{SSR}_0 - \mathrm{SSR}_1)/p}{\mathrm{SSR}_1/(T-2p-1)} \sim F(p, T-2p-1),$$

or

$$pF \sim \chi^2(p).$$

In general, we consider testing the Granger causality from y_j to y_i.

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t.$$

$y_t : k \times 1, \qquad \mu : k \times 1, \qquad \phi_p : k \times k, \qquad \epsilon_t : k \times 1.$

The null hypothesis is: $H_0 : \phi_{ij,1} = \phi_{ij,2} = \cdots = \phi_{ij,p} = 0.$

The alternative hypothesis is: $H_1 :$ not H_0.

SSR_0 = Sum of Squared Residuals under H_0

SSR_1 = Sum of Squared Residuals under H_1

Under H_0, the asymptotic distribution is given by:

$$F = \frac{(\text{SSR}_0 - \text{SSR}_1)/p}{\text{SSR}_1/(T - kp - 1)} \sim F(p, T - kp - 1),$$

or

$$pF \sim \chi^2(p).$$

Example:

Data:　1994 年第一四半期～2014 年第一四半期

`gdp` = GDP (実質，10 億円，季調済，内閣府 HP から取得)

`def` = GDP デフレータ (季調済，内閣府 HP から取得)

`r` = 貸出約定平均金利 (%，新規，総合・国内銀行，日銀 HP から取得)

`m` = 通貨流通高（平均発行高，億円，季調済，日銀 HP から取得)

```
. gen time=_n

. tsset time
        time variable:  time, 1 to 81
                delta:  1 unit

. gen lgdp=log(gdp)

. gen lm=log(m/(def/10))

. varsoc d.lgdp d.r d.lm

  Selection-order criteria
  Sample:  6 - 81                                Number of obs      =        76
 +---------------------------------------------------------------------------+
 |lag |    LL      LR      df    p      FPE       AIC       HQIC      SBIC    |
```

```
 |----+---------------------------------------------------------------------|
 |  0 |   541.22                              1.4e-10  -14.1637  -14.1269  -14.0717  |
 |  1 |  571.181  59.923*    9  0.000  8.2e-11* -14.7153* -14.5682* -14.3473* |
 |  2 |  575.715  9.0675     9  0.431  9.2e-11  -14.5978  -14.3404  -13.9537  |
 |  3 |   579.55  7.6704     9  0.568  1.1e-10  -14.4619  -14.0942  -13.5418  |
 |  4 |  583.767  8.4328     9  0.491  1.2e-10   -14.336   -13.858  -13.1399  |
 +--------------------------------------------------------------------------+
  Endogenous:  D.lgdp D.r D.lm
   Exogenous:  _cons

. var d.lgdp d.r d.lm, lags(1)

Vector autoregression

Sample:  3 - 81                                    No. of obs      =        79
Log likelihood =  592.2334                         AIC             = -14.68945
FPE            =  8.38e-11                         HQIC            = -14.54526
Det(Sigma_ml)  =  6.18e-11                         SBIC            = -14.32954

Equation           Parms      RMSE     R-sq      chi2     P>chi2
----------------------------------------------------------------
D_lgdp                4     .010717   0.0422   3.480972   0.3232
D_r                   4     .087186   0.2553    27.0782   0.0000
D_lm                  4     .009434   0.2903   32.30929   0.0000
----------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
D_lgdp       |
        lgdp |
         LD. |   .2031129   .1119361     1.81   0.070    -.0162778    .4225037
             |
```

r						
LD.	.0045431	.0120151	0.38	0.705	-.0190061	.0280922
lm						
LD.	.0152162	.1086739	0.14	0.889	-.1977807	.228213
_cons	.0019504	.0019124	1.02	0.308	-.0017978	.0056986
D_r						
lgdp						
LD.	.4341641	.9106374	0.48	0.634	-1.350652	2.218981
r						
LD.	.5085677	.0977469	5.20	0.000	.3169874	.7001481
lm						
LD.	.1845222	.8840978	0.21	0.835	-1.548278	1.917322
_cons	-.0202984	.0155578	-1.30	0.192	-.0507912	.0101943
D_lm						
lgdp						
LD.	-.1972406	.098541	-2.00	0.045	-.3903774	-.0041037
r						
LD.	-.029395	.0105773	-2.78	0.005	-.0501261	-.0086639
lm						
LD.	.4472679	.0956691	4.68	0.000	.2597599	.634776
_cons	.0071036	.0016835	4.22	0.000	.0038039	.0104033

```
. vargranger

   Granger causality Wald tests
  +------------------------------------------------------------------+
  |          Equation           Excluded |   chi2     df Prob > chi2 |
  |--------------------------------------+---------------------------|
  |            D_lgdp                D.r |  .14297     1    0.705     |
  |            D_lgdp               D.lm |   .0196     1    0.889     |
  |            D_lgdp                ALL |  .15705     2    0.924     |
  |--------------------------------------+---------------------------|
  |               D_r             D.lgdp |  .22731     1    0.634     |
  |               D_r               D.lm |  .04356     1    0.835     |
  |               D_r                ALL |   .3039     2    0.859     |
  |--------------------------------------+---------------------------|
  |              D_lm             D.lgdp |  4.0064     1    0.045     |
  |              D_lm                D.r |  7.7232     1    0.005     |
  |              D_lm                ALL |  10.798     2    0.005     |
  +------------------------------------------------------------------+
```

8.3 Impulse Response Function (インパルス応答関数):

$$\frac{\partial y_{i,t+m}}{\partial \epsilon_{j,t}}, \qquad m = 1, 2, \cdots,$$

where $i, j = 1, 2, \cdots, k$.

Example: AR(p) Process:

When y_t is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L - \phi_2 L^2 - \ \cdots \ - \phi_p L^p)^{-1} \epsilon_t \\ &= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ \cdots \end{aligned}$$

The impulse response function is:

$$\frac{\partial y_{i,t+k}}{\partial \epsilon_{j,t}} = \theta_{ij,k}, \qquad k = 1, 2, \cdots,$$

where $\theta_{ij,k}$ denotes the (i, j)th element of θ_k.

$$
\begin{aligned}
y_t &= \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots \\
&= PP^{-1}\epsilon_t + \theta_1 PP^{-1}\epsilon_{t-1} + \theta_2 PP^{-1}\epsilon_{t-2} + \cdots \\
&= \Omega_0\eta_t + \Omega_1\eta_{t-1} + \Omega_2\eta_{t-2} + \cdots,
\end{aligned}
$$

where $\mathrm{V}(\eta_t) = I_k$, and $\Omega_i = \theta_i P$ for $i = 0, 1, 2, \cdots$ and $\Omega_0 = P$.

$$
\frac{\partial y_{i,t+m}}{\partial \eta_{j,t}}, \qquad m = 1, 2, \cdots,
$$

where $i, j = 1, 2, \cdots, k$.

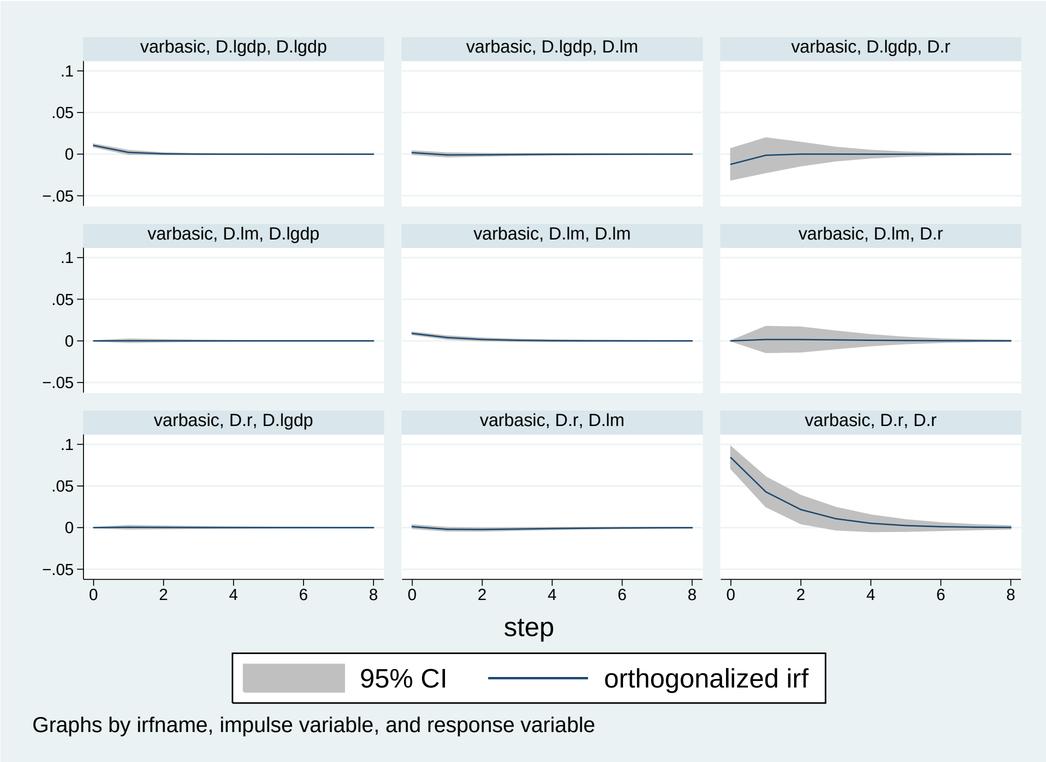
$\Longrightarrow$ **Orthogonalized Impulse Response Function** (直交化インパルス応答関数)

Example:

```
. varbasic d.lgdp d.r d.lm, lags(1)

Vector autoregression
```

```
Sample:  3 - 81                                   No. of obs      =        79
Log likelihood =  592.2334                        AIC             = -14.68945
FPE            =  8.38e-11                        HQIC            = -14.54526
Det(Sigma_ml)  =  6.18e-11                        SBIC            = -14.32954

Equation           Parms      RMSE     R-sq      chi2     P>chi2
----------------------------------------------------------------
D_lgdp                4     .010717   0.0422   3.480972   0.3232
D_r                   4     .087186   0.2553    27.0782   0.0000
D_lm                  4     .009434   0.2903   32.30929   0.0000
----------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
D_lgdp       |
        lgdp |
         LD. |   .2031129   .1119361     1.81   0.070    -.0162778    .4225037
             |
           r |
         LD. |   .0045431   .0120151     0.38   0.705    -.0190061    .0280922
             |
          lm |
         LD. |   .0152162   .1086739     0.14   0.889    -.1977807     .228213
             |
       _cons |   .0019504   .0019124     1.02   0.308    -.0017978    .0056986
-------------+----------------------------------------------------------------
D_r          |
        lgdp |
         LD. |   .4341641   .9106374     0.48   0.634    -1.350652    2.218981
             |
           r |
```

LD.	.5085677	.0977469	5.20	0.000	.3169874	.7001481
lm						
LD.	.1845222	.8840978	0.21	0.835	-1.548278	1.917322
_cons	-.0202984	.0155578	-1.30	0.192	-.0507912	.0101943
D_lm						
lgdp						
LD.	-.1972406	.098541	-2.00	0.045	-.3903774	-.0041037
r						
LD.	-.029395	.0105773	-2.78	0.005	-.0501261	-.0086639
lm						
LD.	.4472679	.0956691	4.68	0.000	.2597599	.634776
_cons	.0071036	.0016835	4.22	0.000	.0038039	.0104033


varbasic, D.lgdp, D.lgdp
varbasic, D.lgdp, D.lm
varbasic, D.lgdp, D.r
varbasic, D.lm, D.lgdp
varbasic, D.lm, D.lm
varbasic, D.lm, D.r
varbasic, D.r, D.lgdp
varbasic, D.r, D.lm
varbasic, D.r, D.r
.1
.05
0
−.05
0
2
4
6
8
step
95% CI
orthogonalized irf
Graphs by irfname, impulse variable, and response variable