Example 5: Multinomial logit model:

The ith individual has $m + 1$ choices, i.e., $j = 0, 1, \cdots, m$.

$$P(y_i = j) = \frac{\exp(X_i\beta_j)}{\sum_{j=0}^{m} \exp(X_i\beta_j)} \equiv P_{ij},$$

for $\beta_0 = 0$. The case of $m = 1$ corresponds to the bivariate logit model (binary choice).

Note that

$$\log \frac{P_{ij}}{P_{i0}} = X_i\beta_j$$

The log-likelihood function is:

$$\log L(\beta_1, \cdots, \beta_m) = \sum_{i=1}^{n} \sum_{j=0}^{m} d_{ij} \ln P_{ij},$$

where $d_{ij} = 1$ when the ith individual chooses jth choice, and $d_{ij} = 0$ otherwise.

Example 6: Nested logit model:

(i) In the 1st step, choose YES or NO. Each probability is P_Y and $P_N = 1 - P_Y$.

(ii) Stop if NO is chosen in the 1st step. Go to the next if YES is chosen in the 1st step.

(iii) In the 2nd step, choose A or B if YES is chosen in the 1st step. Each probability is $P_{A|Y}$ and $P_{B|Y}$.

For simplicity, usually we assume the logistic distribution.

So, we call the nested logit model.

The probability that the ith individual chooses NO is:

$$P_{N,i} = \frac{1}{1 + \exp(X_i\beta)}.$$

The probability that the ith individual chooses YES and A is:

$$P_{A|Y,i}P_{Y,i} = P_{A|Y,i}(1 - P_{N,i}) = \frac{\exp(Z_i\alpha)}{1 + \exp(Z_i\alpha)}\frac{\exp(X_i\beta)}{1 + \exp(X_i\beta)}.$$

The probability that the ith individual chooses YES and B is:

$$P_{B|Y,i}P_{Y,i} = (1 - P_{A|Y,i})(1 - P_{N,i}) = \frac{1}{1+\exp(Z_i\alpha)}\frac{\exp(X_i\beta)}{1+\exp(X_i\beta)}.$$

In the 1st step, decide if the ith individual buys a car or not.

In the 2nd step, choose A or B.

X_i includes annual income, distance from the nearest station, and so on.

Z_i are speed, fuel-efficiency, car company, color, and so on.

The likelihood function is:

$$\begin{aligned} L(\alpha,\beta) &= \prod_{i=1}^{n} P_{N,i}^{I_{1i}}\Big(\big((1-P_{N,i})P_{A|Y,i}\big)^{I_{2i}}\big((1-P_{N,i})(1-P_{A|Y,i})\big)^{1-I_{2i}}\Big)^{1-I_{1i}} \\ &= \prod_{i=1}^{n} P_{N,i}^{I_{1i}}(1-P_{N,i})^{1-I_{1i}}\Big(P_{A|Y,i}^{I_{2i}}(1-P_{A|Y,i})^{1-I_{2i}}\Big)^{1-I_{1i}}, \end{aligned}$$

where

$$I_{1i} = \begin{cases} 1, & \text{if the } i\text{th individual decides not to buy a car in the 1st step,} \\ 0, & \text{if the } i\text{th individual decides to buy a car in the 1st step,} \end{cases}$$

$$I_{2i} = \begin{cases} 1, & \text{if the } i\text{th individual chooses A in the 2nd step,} \\ 0, & \text{if the } i\text{th individual chooses B in the 2nd step,} \end{cases}$$

Remember that $\mathrm{E}(y_i) = F(X_i\beta^*)$, where $\beta^* = \dfrac{\beta}{\sigma}$.
Therefore, size of β^* does not mean anything.

The marginal effect is given by:

$$\frac{\partial \mathrm{E}(y_i)}{\partial X_i} = f(X_i\beta^*)\beta^*.$$

Thus, the marginal effect depends on the height of the density function $f(X_i\beta^*)$.

2.2 Limited Dependent Variable Model (制限従属変数モデル)

Truncated Regression Model: Consider the following model:

$$y_i = X_i\beta + u_i, \qquad u_i \sim N(0, \sigma^2) \text{ when } y_i > a, \text{ where } a \text{ is a constant,}$$

for $i = 1, 2, \cdots, n$.

Consider the case of $y_i > a$ (i.e., in the case of $y_i \le a$, y_i is not observed).

$$\mathrm{E}(u_i | X_i\beta + u_i > a) = \int_{a - X_i\beta}^{\infty} u_i \frac{f(u_i)}{1 - F(a - X_i\beta)} \mathrm{d}u_i.$$

Suppose that $u_i \sim N(0, \sigma^2)$, i.e., $\dfrac{u_i}{\sigma} \sim N(0, 1)$.

Using the following standard normal density and distribution functions:

$$\phi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2),$$

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-\frac{1}{2}z^2)\mathrm{d}z = \int_{-\infty}^{x} \phi(z)\mathrm{d}z,$$