

- However,  $\beta_{MM}$  is inconsistent when  $E(x'u) \neq 0$ , i.e.,

$$\beta_{MM} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u = \beta + \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'u\right) \not\rightarrow \beta.$$

Note as follows:

$$\frac{1}{n}X'u = \frac{1}{n} \sum_{i=1}^n x_i'u_i \longrightarrow E(x'u) \neq 0.$$

In order to obtain a consistent estimator of  $\beta$ , we find the instrumental variable  $z$  which satisfies  $E(z'u) = 0$ .

Let  $z_i$  be the  $i$ th realization of  $z$ , where  $z_i$  is a  $1 \times k$  vector.

Then, we have the following:

$$\frac{1}{n}Z'u = \frac{1}{n} \sum_{i=1}^n z_i'u_i \longrightarrow E(z'u) = 0.$$

The  $\beta$  which satisfies  $\frac{1}{n} \sum_{i=1}^n z_i'u_i = 0$  is denoted by  $\beta_{IV}$ , i.e.,  $\frac{1}{n} \sum_{i=1}^n z_i'(y_i - x_i\beta_{IV}) = 0$ .

Thus,  $\beta_{IV}$  is obtained as:

$$\beta_{IV} = \left( \frac{1}{n} \sum_{i=1}^n z_i' x_i \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^n z_i' y_i \right) = (Z'X)^{-1} Z'y.$$

Note that  $Z'X$  is a  $k \times k$  square matrix, where we assume that the inverse matrix of  $Z'X$  exists.

Assume that as  $n$  goes to infinity there exist the following moment matrices:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n z_i' x_i &= \frac{1}{n} Z'X \longrightarrow M_{zx}, \\ \frac{1}{n} \sum_{i=1}^n z_i' z_i &= \frac{1}{n} Z'Z \longrightarrow M_{zz}, \\ \frac{1}{n} \sum_{i=1}^n z_i' u_i &= \frac{1}{n} Z'u \longrightarrow 0. \end{aligned}$$

As  $n$  goes to infinity,  $\beta_{IV}$  is rewritten as:

$$\begin{aligned}\beta_{IV} &= (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u \\ &= \beta + \left(\frac{1}{n}Z'X\right)^{-1}\left(\frac{1}{n}Z'u\right) \longrightarrow \beta + M_{zx} \times 0 = \beta,\end{aligned}$$

Thus,  $\beta_{IV}$  is a consistent estimator of  $\beta$ .

- We consider the asymptotic distribution of  $\beta_{IV}$ .

By the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz})$$

$$\begin{aligned}\text{Note that } V\left(\frac{1}{\sqrt{n}}Z'u\right) &= \frac{1}{n}V(Z'u) = \frac{1}{n}E(Z'uu'Z) = \frac{1}{n}E\left(E(Z'uu'Z|Z)\right) \\ &= \frac{1}{n}E\left(Z'E(uu'|Z)Z\right) = \frac{1}{n}E(\sigma^2 Z'Z) = E\left(\sigma^2 \frac{1}{n}Z'Z\right) \longrightarrow E(\sigma^2 M_{zz}) = \sigma^2 M_{zz}.\end{aligned}$$

We obtain the following asymptotic distribution:

$$\sqrt{n}(\beta_{IV} - \beta) = \left(\frac{1}{n}Z'X\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}^{-1'})$$

Practically, for large  $n$  we use the following distribution:

$$\beta_{IV} \sim N\left(\beta, s^2(Z'X)^{-1}Z'Z(Z'X)^{-1'}\right),$$

where  $s^2 = \frac{1}{n-k}(y - X\beta_{IV})'(y - X\beta_{IV})$ .

- In the case where  $z_i$  is a  $1 \times r$  vector for  $r > k$ ,  $Z'X$  is a  $r \times k$  matrix, which is not a square matrix.  $\implies$  **Generalized Method of Moments (GMM, 一般化積率法)**

## 5.2 Generalized Method of Moments (GMM, 一般化積率法)

In order to obtain a consistent estimator of  $\beta$ , we have to find the instrumental variable  $z$  which satisfies  $E(z'u) = 0$ .

For now, however, suppose that we have  $z$  with  $E(z'u) = 0$ .

Let  $z_i$  be the  $i$ th realization (i.e., the  $i$ th data) of  $z$ , where  $z_i$  is a  $1 \times r$  vector and  $r > k$ .

Then, using the law of large number, we have the following:

$$\frac{1}{n}Z'u = \frac{1}{n} \sum_{i=1}^n z_i'u_i = \frac{1}{n} \sum_{i=1}^n z_i'(y_i - x_i\beta) \longrightarrow E(z'u) = 0.$$

The number of equations (i.e.,  $r$ ) is larger than the number of parameters (i.e.,  $k$ ).

Let us define  $W$  as a  $r \times r$  weight matrix, which is symmetric.

We solve the following minimization problem:

$$\min_{\beta} \left( \frac{1}{n} \sum_{i=1}^n z'_i (y_i - x_i \beta) \right)' W \left( \frac{1}{n} \sum_{i=1}^n z'_i (y_i - x_i \beta) \right),$$

which is equivalent to:

$$\min_{\beta} (Z'(y - X\beta))' W (Z'(y - X\beta)),$$

i.e.,

$$\min_{\beta} (y - X\beta)' ZWZ'(y - X\beta).$$

Note that  $\sum_{i=1}^n z'_i (y_i - x_i \beta) = Z'(y - X\beta)$ .

$W$  should be the inverse matrix of the variance-covariance matrix of  $Z'(y - X\beta) = Z'u$ .

Suppose that  $V(u) = \sigma^2 \Omega$ .

Then,  $V(Z'u) = E(Z'u(Z'u)') = E(Z'uu'Z) = Z'E(uu')Z = \sigma^2 Z'\Omega Z = W^{-1}$ .

The following minimization problem should be solved.

$$\min_{\beta} (y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta).$$

The solution of  $\beta$  is given by the GMM estimator, denoted by  $\beta_{GMM}$ .

**Remark:** For the model:  $y = X\beta + u$  and  $u \sim (0, \sigma^2\Omega)$ , solving the following minimization problem:

$$\min_{\beta} (y - X\beta)'\Omega^{-1}(y - X\beta),$$

GLS is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

Note that  $b$  is the best linear unbiased estimator.