Serially Correlated Errors (Time Series Data):

• Suppose that $u_1, u_2, \cdots, u_n$ are serially correlated.

Consider the case where the subscript represents time.

Remember that $\beta_{GMM} \sim N\big(\beta, \, \sigma^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}\big)$,

We need to consider evaluation of $\sigma^2 Z'\Omega Z = V(u^*)$, i.e.,

$$
\begin{aligned}
V(u^*) = V(Z'u) &= V(\sum_{i=1}^{n} z_i'u_i) = V(\sum_{i=1}^{n} v_i) \\
&= E\big((\sum_{i=1}^{n} v_i)(\sum_{i=1}^{n} v_i)'\big) = E\big((\sum_{i=1}^{n} v_i)(\sum_{j=1}^{n} v_j)'\big) \\
&= E(\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j') = \sum_{i=1}^{n} \sum_{j=1}^{n} E(v_i v_j')
\end{aligned}
$$

where $v_i = z_i'u_i$ is a $r \times 1$ vector.

Define $\Gamma_\tau = E(v_i v'_{i-\tau})$.

$\Gamma_0 = E(v_i v'_i)$ represents the $r \times r$ variance-covariance matrix of v_i.

$\Gamma_{-\tau} = E(v_{i-\tau} v'_i) = E((v_i v'_{i-\tau})') = \left(E(v_i v'_{i-\tau})\right)' = \Gamma'_\tau.$

$$
\begin{aligned}
V(u^*) &= \sum_{i=1}^{n} \sum_{j=1}^{n} E(v_i v'_j) \\
&= E(v_1 v'_1) + E(v_1 v'_2) + E(v_1 v'_3) + \cdots + E(v_1 v'_n) \\
&\quad + E(v_2 v'_1) + E(v_2 v'_2) + E(v_2 v'_3) + \cdots + E(v_2 v'_n) \\
&\quad + E(v_3 v'_1) + E(v_3 v'_2) + E(v_3 v'_3) + \cdots + E(v_3 v'_n) \\
&\qquad\qquad \vdots \\
&\quad + E(v_n v'_1) + E(v_n v'_2) + E(v_n v'_3) + \cdots + E(v_n v'_n) \\
&= \Gamma_0 + \Gamma_{-1} + \Gamma_{-2} + \cdots + \Gamma_{1-n} \\
&\quad + \Gamma_1 + \Gamma_0 + \Gamma_{-1} + \cdots + \Gamma_{2-n}
\end{aligned}
$$

$$+ \Gamma_2 + \Gamma_1 + \Gamma_0 + \cdots + \Gamma_{3-n}$$
$$\vdots$$
$$+ \Gamma_{n-1} + \Gamma_{n-2} + \Gamma_{n-3} + \cdots + \Gamma_0$$

$$= \Gamma_0 + \Gamma_1' + \Gamma_2' + \cdots + \Gamma_{n-1}'$$
$$+ \Gamma_1 + \Gamma_0 + \Gamma_1' + \cdots + \Gamma_{n-2}'$$
$$+ \Gamma_2 + \Gamma_1 + \Gamma_0 + \cdots + \Gamma_{n-3}'$$
$$\vdots$$
$$+ \Gamma_{n-1} + \Gamma_{n-2} + \Gamma_{n-3} + \cdots + \Gamma_0$$

$$= n\Gamma_0 + (n-1)(\Gamma_1 + \Gamma_1') + (n-2)(\Gamma_2 + \Gamma_2') + \cdots + (\Gamma_{n-1} + \Gamma_{n-1}')$$
$$= n\Gamma_0 + \sum_{i=1}^{n-1}(n-i)(\Gamma_i + \Gamma_i')$$

$$= n\left(\Gamma_0 + \sum_{i=1}^{n-1}(1 - \frac{i}{n})(\Gamma_i + \Gamma_i')\right)$$

$$\approx n\left(\Gamma_0 + \sum_{i=1}^{q}(1 - \frac{i}{q+1})(\Gamma_i + \Gamma_i')\right).$$

In the last line, $n - 1$ is replaced by q, where $q < n - 1$.

We need to estimate Γ_τ as: $\quad \hat{\Gamma}_\tau = \frac{1}{n}\sum_{i=\tau+1}^{n}\hat{v}_i\hat{v}_{i-\tau}'$, where $\hat{v}_i = z_i'\hat{u}_i$ for $\hat{u}_i = y_i - x_i\beta_{GMM}$.

As τ is large, $\hat{\Gamma}_\tau$ is unstable.

Therefore, we choose the q which is less than $n - 1$.