Econometrics 2 (2018) TA session 11*

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20 December 2018

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5 GMM (generalized method of moments)

5.1 GMM: linear regression

When $E[u|X] \neq 0, E[u|Z] = 0, X : (n \times k), Z : (n \times r), r \geq k$,
consider the following linear regression model

$$y = X\beta + u.$$

In this case, multiply Z from left hand side and rewrite the equation as

$$y^* = X^*\beta + u^*, \quad u^* \sim (0, \sigma^2 Z'Z)$$

where $y^* = Z'y, X^* = Z'X, u^* = Z'u$. The GMM estimator can be derived by solving the following minimizing problem,

$$\min_{\beta} u^*(Z'Z)^{-1}u^*.$$

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* All comments welcome!

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From FOC,

$$\frac{\partial u^*(Z'Z)^{-1}u^*}{\partial \beta} = 0$$
$$\Longrightarrow \hat{\beta}_{\text{GMM}} = (X'P_Z X)^{-1}X'P_Z y.$$

where $P_Z = Z(Z'Z)^{-1}Z'$.

5.2 GMM: non-linear case

Consider the case that the moment condition is non-linear;

$$E[h(\theta : w_i) = 0,$$

where θ is $k \times 1$ parameter vactor, $w_i = (y_i, x_i), i = 1, \cdots, n$.
Similarly, we can derive GMM estimator by solving follwing minimizing problem

$$\min_{\theta} q \equiv \bar{g}_n' S^{-1} \bar{g}_n,$$

where $\bar{g}_n(\theta : W) = \frac{1}{n}\sum_{i=1}^n h(\theta : w_i)$, S is symmetric matrix. From FOC,

$$\frac{\partial q}{\partial \theta} = 2\frac{\partial \bar{g}_n(\theta : W)}{\partial \theta} S^{-1}\bar{g}_n = 0$$

To obtain $\hat{\theta}_{\text{GMM}}$, linearize the first-order condition around $\theta = \hat{\theta}_{\text{GMM}}$,

$$\begin{aligned}
0 &= \frac{\partial \bar{g}_n(\hat{\theta}_{\text{GMM}} : W)}{\partial \theta} s^{-1}\bar{g}_n(\hat{\theta}_{\text{GMM}} : W) \\
&\approx \frac{\partial \bar{g}_n(\theta : W)}{\partial \theta} S^{-1}(\bar{g}_n(\theta : W) + \frac{\partial \bar{g}_n(\theta : W)}{\partial \theta'}(\hat{\theta}_{\text{GMM}} - \theta) \\
\Longleftrightarrow \hat{\theta}_{\text{GMM}} &= \theta - \left(\frac{\partial \bar{g}_n(\theta : W)}{\partial \theta} S^{-1} \frac{\partial \bar{g}_n(\theta : W)}{\partial \theta'}\right)^{-1} \frac{\partial \bar{g}_n(\theta : W)}{\partial \theta} S^{-1}\bar{g}_n(\theta : W)
\end{aligned}$$

Replacing θ and $\hat{\theta}_{\text{GMM}}$ by $\hat{\theta}^{i+1}$ and $\hat{\theta}^i$,

$$\hat{\theta}^{i+1} = \hat{\theta}^i - (\hat{D}_i S^{-1}\hat{D}_i')^{-1}\hat{D}_i S{-}1\bar{g}_n(\hat{\theta}^i : W)$$

where $\hat{D}_i \equiv \frac{\bar{g}_n(\hat{\theta}^i : W)}{\partial \theta}$.

5.2.1 How to estimate $\hat{S}$?

In this case, S is the variance of $\sqrt{n}\bar{g}_n(\theta : W)$;

$$
\begin{aligned}
S = V[\sqrt{n}\bar{g}_n(\theta : W)] &= \frac{1}{n} V\left[\sum_{i=1}^{n} h(\theta : w_i)\right] \\
&= \frac{1}{n} E\left[\sum_{i=1}^{n} h(\theta : w_i) \sum_{i=1}^{n} h(\theta : w_i)'\right] \\
&= \frac{1}{n}\{n\Gamma_0 + (n-1)(\Gamma_1 + \Gamma_1') + (n-2)(\Gamma_2 + \Gamma_2') \cdots + (\Gamma_{n-1} + \Gamma_{n-1})\} \\
&= \Gamma_0 + \sum_{i=1}^{n-1}\left(1 - \frac{i}{n}\right)(\Gamma_i + \Gamma_i')
\end{aligned}
$$

where $\Gamma_\tau = E[h(\theta : w_i)h(\theta : w_{i-\tau})']$. The estimator of S is

$$
\hat{S} = \hat{\Gamma}_0 + \sum_{i=1}^{q-1}\left(1 - \frac{i}{q+1}\right)(\hat{\Gamma}_i + \hat{\Gamma}_i')
$$

where $q \leq n$

$$
\hat{\Gamma}_\tau = \frac{1}{n}\sum_{i=\tau+1}^{n} h(\theta : w_i)h(\theta : w_{i-\tau})'
$$

5.3 Testing hypothesis

In this subsection, we assume that

1. $\hat{\theta}_{\text{GMM}} \to \theta$
2. $\sqrt{n}\bar{g}_n(\theta : W) \to N(0, S),\ S = \lim_{n\to\infty} V[\sqrt{n}\bar{g}_n(\theta : W)]$

5.3.1 Asymptotic distribution of GMM estimator

$\hat{\theta}_{GMM}$ satisfy

$$
q \equiv \hat{D}'\hat{S}^{-1}\bar{g}_n(\hat{\theta}_{\text{GMM}} : W) = 0 \tag{1}
$$

where

$$
\hat{D}' \equiv \frac{\partial \bar{g}_n(\hat{\theta}_{\text{GMM}} : W)'}{\partial \theta}.
$$

Linearize $\bar{g}_n(\hat{\theta}_{\text{GMM}} : W)$ around $\hat{\theta}_{\text{GMM}} = \theta$ as follows:

$$\bar{g}_n(\hat{\theta}_{\text{GMM}}) = \bar{g}_n(\theta : W) + \frac{\partial \bar{g}_n(\bar{\theta} : W)}{\partial \theta'}(\hat{\theta}_{\text{GMM}} - \theta)$$

where $\bar{\theta}$ is between $\hat{\theta}_{\text{GMM}}$ and θ. (1) can be rewritten as

$$0 = \hat{D}'\hat{S}^{-1}(\bar{g}_n(\theta : W) + \bar{D}(\hat{\theta}_{\text{GMM}} - \theta))$$

where

$$\underset{(r\times k)}{\bar{D}} = \frac{\partial \bar{g}_n(\bar{\theta} : W)}{\partial \theta'}.$$

By using assumption 2, asymptotic distribution is

$$\begin{aligned}\sqrt{n}(\hat{\theta}_{\text{GMM}} - \theta) &= (\hat{D}'\hat{S}^{-1}\bar{D})^{-1}\hat{D}'\hat{S}^{-1} \cdot \sqrt{n}\bar{g}_n(\theta : W) \\ &\to N(0, (DS^{-1}D)^{-1})\end{aligned}$$

where $\hat{D} \to D, \bar{D} \to D, \hat{S} \to S$ because of $\hat{\theta}_{\text{GMM}} \to \theta, \bar{\theta} \to \theta$.

5.3.2 Testing hypothesis

In this subsection, we consider the following hypothesis

- $H_0 : \underset{(p\times 1)}{R(\theta)} = 0$
- $H_1 : R(\theta) \neq 0$

By delta method,

$$R(\hat{\theta}_{\text{GMM}}) = R(\theta) + R_{\bar{\theta}}(\hat{\theta}_{\text{GMM}} - \theta)$$

where

$$R_{\bar{\theta}} \equiv \frac{\partial R(\bar{\theta})}{\partial \theta'},$$

$\bar{\theta}$ is between θ and $\hat{\theta}_{\text{GMM}}$. Asymptotic distribution of $\sqrt{n}(R(\hat{\theta}_{\text{GMM}}) - R(\theta))$ is

$$\begin{aligned}\sqrt{n}(R(\hat{\theta}_{\text{GMM}}) - R(\theta)) &= R_{\bar{\theta}}\sqrt{n}(\hat{\theta}_{\text{GMM}} - \theta) \\ &\to N(0, R_{\theta}(D'S^{-1}D)^{-1}R'_{\theta})\end{aligned}$$

because $R_{\bar{\theta}} \to R_{\theta}$ as $\hat{\theta}_{\text{GMM}} \to \theta$. So we have following distribution.

$$n \cdot (R(\hat{\theta}_{\text{GMM}}) - R(\theta))'(R_{\hat{\theta}_{\text{GMM}}}(\hat{D}'\hat{S}^{-1}\hat{D})^{-1})(R(\hat{\theta}_{\text{GMM}}) - R(\theta)) \to \chi^2(p)$$

Under $H_0 : R(\theta) = 0$, the test statistic is

$$n \cdot (R(\hat{\theta}_{\text{GMM}}))'(R_{\hat{\theta}_{\text{GMM}}}(\hat{D}'\hat{S}^{-1}\hat{D})^{-1})(R(\hat{\theta}_{\text{GMM}})) \to \chi^2(p)$$