# 3 Panel Data

## 3.1 GLS — Review

**Regression model I:**

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I_n),$$

where $y$, $X$, $\beta$, $u$, 0 and $I_n$ are $n \times 1$, $n \times k$, $k \times 1$, $n \times 1$, $n \times 1$, and $n \times n$, respectively.

We solve the following minimization problem:

$$\min_{\beta} (y - X\beta)'(y - X\beta).$$

Let $\hat{\beta}$ be a solution of the above minimization problem.

OLS estimator of $\beta$ is given by:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u.$$

$$E(\hat{\beta}) = \beta, \qquad V(\hat{\beta}) = \sigma^2(X'X)^{-1}.$$

### Regression model II:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega),$$

where $\Omega$ is $n \times n$.

We solve the following minimization problem:

$$\min_{\beta} (y - X\beta)'\Omega^{-1}(y - X\beta).$$

Let $b$ be a solution of the above minimization problem.

GLS estimator of $\beta$ is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u.$$

$$E(b) = \beta, \qquad V(b) = \sigma^2(X'\Omega^{-1}X)^{-1}.$$

• We apply OLS to the following regression model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

OLS estimator of $\beta$ is given by:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u.$$

$$E(\hat{\beta}) = \beta, \qquad V(\hat{\beta}) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}.$$

$\hat{\beta}$ is an unbiased estimator.

The difference between two variances is:

$$
\begin{aligned}
&\mathrm{V}(\hat{\beta}) - \mathrm{V}(b) \\
&= \sigma^2 (X'X)^{-1} X' \Omega X (X'X)^{-1} - \sigma^2 (X'\Omega^{-1}X)^{-1} \\
&= \sigma^2 \big( (X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1} \big) \Omega \big( (X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1} \big)' \\
&= \sigma^2 A\Omega A'
\end{aligned}
$$

$\Omega$ is the variance-covariance matrix of $u$, which is a positive definite matrix.
Therefore, except for $\Omega = I_n$, $A\Omega A'$ is also a positive definite matrix.

This implies that $\mathrm{V}(\hat{\beta}_i) - \mathrm{V}(b_i) > 0$ for the $i$th element of $\beta$.
Accordingly, $b$ is more efficient than $\hat{\beta}$.

## 3.2 Panel Model Basic

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where $i$ indicates individual and $t$ denotes time.

There are $n$ observations for each $t$.

$u_{it}$ indicates the error term, assuming that $E(u_{it}) = 0$, $V(u_{it}) = \sigma_u^2$ and $Cov(u_{it}, u_{js}) = 0$ for $i \neq j$ and $t \neq s$.

$v_i$ denotes the individual effect, which is fixed or random.

### 3.2.1 Fixed Effect Model (固定効果モデル)

In the case where $v_i$ is fixed, the case of $v_i = z_i\alpha$ is included.

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

$$\overline{y}_i = \overline{X}_i\beta + v_i + \overline{u}_i, \qquad i = 1, 2, \cdots, n,$$

where $\overline{y}_i = \dfrac{1}{T} \sum_{t=1}^{T} y_{it}$, $\overline{X}_i = \dfrac{1}{T} \sum_{t=1}^{T} X_{it}$, and $\overline{u}_i = \dfrac{1}{T} \sum_{t=1}^{T} u_{it}$.

$$(y_{it} - \overline{y}_i) = (X_{it} - \overline{X}_i)\beta + (u_{it} - \overline{u}_i), \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

Taking an example of $y$, the left-hand side of the above equation is rewritten as:

$$y_{it} - \overline{y}_i = y_{it} - \frac{1}{T}1_T' y_i,$$

where $1_T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, which is a $T \times 1$ vector, and $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$.

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = I_T y_i - 1_T \overline{y}_i = I_T y_i - \frac{1}{T} 1_T 1_T' y_i = (I_T - \frac{1}{T} 1_T 1_T') y_i$$

Thus,

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = \begin{pmatrix} X_{i1} - \overline{X}_i \\ X_{i2} - \overline{X}_i \\ \vdots \\ X_{iT} - \overline{X}_i \end{pmatrix} \beta + \begin{pmatrix} u_{i1} - \overline{u}_i \\ u_{i2} - \overline{u}_i \\ \vdots \\ u_{iT} - \overline{u}_i \end{pmatrix}, \qquad i = 1, 2, \cdots, n,$$

which is re-written as:

$$(I_T - \frac{1}{T}1_T 1_T')y_i = (I_T - \frac{1}{T}1_T 1_T')X_i\beta + (I_T - \frac{1}{T}1_T 1_T')u_i, \qquad i = 1, 2, \cdots, n,$$

i.e.,

$$D_T y_i = D_T X_i \beta + D_T u_i, \qquad i = 1, 2, \cdots, n,$$

where $D_T = (I_T - \frac{1}{T}1_T 1_T')$, which is a $T \times T$ matrix.

Note that $D_T D_T' = D_T$, i.e., $D_T$ is a symmetric and idempotent matrix.

Using the matrix form for $i = 1, 2, \cdots, n$, we have:

$$\begin{pmatrix} D_T y_1 \\ D_T y_2 \\ \vdots \\ D_T y_n \end{pmatrix} = \begin{pmatrix} D_T X_1 \\ D_T X_2 \\ \vdots \\ D_T X_n \end{pmatrix} \beta + \begin{pmatrix} D_T u_1 \\ D_T u_2 \\ \vdots \\ D_T u_n \end{pmatrix},$$

i.e.,

$$
\begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} y = \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} X\beta + \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} u,
$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $X \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, which are $Tn \times 1$, $Tn \times k$ and $Tn \times 1$ matrices, respectively

Using the Kronecker product, we obtain the following expression:

$$
(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,
$$

where $(I_n \otimes D_T)$, $y$, $X$, and $u$ are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

**Kronecker Product — Review:**

1. $A$: $n \times m$,    $B$: $T \times k$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}, \text{ which is a } nT \times mk \text{ matrix.}$$

2. $A$: $n \times n$,    $B$: $m \times m$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad |A \otimes B| = |A|^m |B|^n,$$

$$(A \otimes B)' = A' \otimes B', \qquad \text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B).$$

3. For $A$, $B$, $C$ and $D$ such that the products are defined,

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

**End of Review**