Serially Correlated Errors (Time Series Data):

- Suppose that $u_1, u_2, \cdots, u_n$ are serially correlated.

Consider the case where the subscript represents time.

Remember that $\beta_{GMM} \sim N\Big(\beta,\ \sigma^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}\Big)$,

We need to consider evaluation of $\sigma^2 Z'\Omega Z = \mathrm{V}(u^*)$, i.e.,

$$\begin{aligned}
\mathrm{V}(u^*) &= \mathrm{V}(Z'u) = \mathrm{V}(\sum_{i=1}^{n} z_i' u_i) = \mathrm{V}(\sum_{i=1}^{n} v_i) \\
&= \mathrm{E}\Big((\sum_{i=1}^{n} v_i)(\sum_{i=1}^{n} v_i)'\Big) = \mathrm{E}\Big((\sum_{i=1}^{n} v_i)(\sum_{j=1}^{n} v_j)'\Big) \\
&= \mathrm{E}(\sum_{i=1}^{n}\sum_{j=1}^{n} v_i v_j') = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}(v_i v_j')
\end{aligned}$$

where $v_i = z_i' u_i$ is a $r \times 1$ vector.

Define $\Gamma_\tau = \mathrm{E}(v_i v'_{i-\tau})$.

$\Gamma_0 = \mathrm{E}(v_i v'_i)$ represents the $r \times r$ variance-covariance matrix of v_i.

$\Gamma_{-\tau} = \mathrm{E}(v_{i-\tau} v'_i) = \mathrm{E}((v_i v'_{i-\tau})') = \left(\mathrm{E}(v_i v'_{i-\tau})\right)' = \Gamma'_\tau$.

$$
\begin{aligned}
\mathrm{V}(u^*) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{E}(v_i v'_j) \\
&= \mathrm{E}(v_1 v'_1) + \mathrm{E}(v_1 v'_2) + \mathrm{E}(v_1 v'_3) + \cdots + \mathrm{E}(v_1 v'_n) \\
&+ \mathrm{E}(v_2 v'_1) + \mathrm{E}(v_2 v'_2) + \mathrm{E}(v_2 v'_3) + \cdots + \mathrm{E}(v_2 v'_n) \\
&+ \mathrm{E}(v_3 v'_1) + \mathrm{E}(v_3 v'_2) + \mathrm{E}(v_3 v'_3) + \cdots + \mathrm{E}(v_3 v'_n) \\
&\qquad \vdots \\
&+ \mathrm{E}(v_n v'_1) + \mathrm{E}(v_n v'_2) + \mathrm{E}(v_n v'_3) + \cdots + \mathrm{E}(v_n v'_n) \\
&= \Gamma_0 + \Gamma_{-1} + \Gamma_{-2} + \cdots + \Gamma_{1-n} \\
&+ \Gamma_1 + \Gamma_0 + \Gamma_{-1} + \cdots + \Gamma_{2-n}
\end{aligned}
$$

$$+ \Gamma_2 + \Gamma_1 + \Gamma_0 + \cdots + \Gamma_{3-n}$$

$$\vdots$$

$$+ \Gamma_{n-1} + \Gamma_{n-2} + \Gamma_{n-3} + \cdots + \Gamma_0$$

$$= \Gamma_0 + \Gamma'_1 + \Gamma'_2 + \cdots + \Gamma'_{n-1}$$

$$+ \Gamma_1 + \Gamma_0 + \Gamma'_1 + \cdots + \Gamma'_{n-2}$$

$$+ \Gamma_2 + \Gamma_1 + \Gamma_0 + \cdots + \Gamma'_{n-3}$$

$$\vdots$$

$$+ \Gamma_{n-1} + \Gamma_{n-2} + \Gamma_{n-3} + \cdots + \Gamma_0$$

$$= n\Gamma_0 + (n-1)(\Gamma_1 + \Gamma'_1) + (n-2)(\Gamma_2 + \Gamma'_2) + \cdots + (\Gamma_{n-1} + \Gamma'_{n-1})$$

$$= n\Gamma_0 + \sum_{i=1}^{n-1}(n-i)(\Gamma_i + \Gamma'_i)$$

$$= n\Big(\Gamma_0 + \sum_{i=1}^{n-1}(1 - \frac{i}{n})(\Gamma_i + \Gamma_i')\Big)$$

$$\approx n\Big(\Gamma_0 + \sum_{i=1}^{q}(1 - \frac{i}{q+1})(\Gamma_i + \Gamma_i')\Big).$$

In the last line, $n-1$ is replaced by q, where $q < n-1$.

We need to estimate Γ_τ as: $\hat{\Gamma}_\tau = \frac{1}{n}\sum_{i=\tau+1}^{n} \hat{v}_i \hat{v}_{i-\tau}'$, where $\hat{v}_i = z_i' \hat{u}_i$ for $\hat{u}_i = y_i - x_i \beta_{GMM}$.

As τ is large, $\hat{\Gamma}_\tau$ is unstable.

Therefore, we choose the q which is less than $n-1$.

Hansen's J Test: Is the model specification correct?

That is, is $\mathrm{E}(z'u) = 0$ for $y = x\beta + u$ correct?

H_0 : $\mathrm{E}(z'u) = 0$ (The model is correct. Or, the instrumental variables are appropriate.)

H_1 : $\mathrm{E}(z'u) \neq 0$

The number of equations is r, while the number of parameters is k.

The degree of freedom is $r - k$.

$$\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i'\hat{u}_i\Big)'\Big(\widehat{\mathrm{V}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\hat{u}_i)}\Big)^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i'\hat{u}_i\Big) \longrightarrow \chi(r-k),$$

where $\hat{u}_i = y_i - x_i\beta_{GMM}$.

$\widehat{\mathrm{V}(\frac{1}{n}\sum_{i=1}^{n} z_i'\hat{u}_i)}$ indicates the estimate of $\mathrm{V}(\frac{1}{n}\sum_{i=1}^{n} z_i'u_i)$ for $u_i = y_i - x_i\beta$.

The J test is called a test for over-identifying restrictions (過剰識別制約).

Remark 1: $X_1, X_2, \cdots, X_n$ are mutually independent.

$X_i \sim N(\mu, \sigma^2)$ are assumed.

Consider $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

Then, $\dfrac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0,1)$.

That is, $\sqrt{n}(\overline{X} - \mu) \longrightarrow N(0, \sigma^2)$.

Remark 2: $X_1, X_2, \cdots, X_n$ are mutually independent.

$X_i \sim N(\mu, \sigma^2)$ are assumed.

Then, $\left(\dfrac{X_i - \mu}{\sigma^2}\right)^2 \sim \chi^2(1)$ and $\displaystyle\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma^2}\right)^2 \sim \chi^2(n)$.

If μ is replaced by its estimator $\overline{X}$, then $\displaystyle\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma^2}\right)^2 \sim \chi^2(n-1)$.

Note:

$$\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma^2}\right)^2 = \begin{pmatrix} X_i - \overline{X} \\ X_i - \overline{X} \\ \vdots \\ X_n - \overline{X} \end{pmatrix}' \begin{pmatrix} \sigma^2 & & & 0 \\ & \sigma^2 & & \\ & & \ddots & \\ 0 & & & \sigma^2 \end{pmatrix}^{-1} \begin{pmatrix} X_i - \overline{X} \\ X_i - \overline{X} \\ \vdots \\ X_n - \overline{X} \end{pmatrix} \sim \chi^2(n-1)$$

In the case of GMM,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' u_i \longrightarrow N(0,\Sigma),$$

where $\Sigma = \mathrm{V}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' u_i\Big)$.

Therefore, we obtain: $\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' u_i\Big)'\Sigma^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' u_i\Big) \longrightarrow \chi^2(r)$.

In order to obtain $\hat{u}_i$, we have to estimate β, which is a $k \times 1$ vector.

Therefore, replacing u_i by $\hat{u}_i$, we have: $\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' \hat{u}_i\Big)'\Sigma^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' \hat{u}_i\Big) \longrightarrow \chi^2(r-k)$.

Moreover, from $\hat{\Sigma} \longrightarrow \Sigma$, we obtain: $\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' \hat{u}_i\Big)'\hat{\Sigma}^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i' \hat{u}_i\Big) \longrightarrow \chi^2(r-k)$,

where $\hat{\Sigma}$ is a consistent estimator of Σ.

4.3 Generalized Method of Moments (GMM, 一般化積率法) II — Nonlinear Case —

Consider the general case:

$$\mathrm{E}(h(\theta; w)) = 0,$$

which is the orthogonality condition.

A $k \times 1$ vector θ denotes a parameter to be estimated.

$h(\theta; w)$ is a $r \times 1$ vector for $r \geq k$.

Let $w_i = (y_i, x_i)$ be the ith observed data, i.e., the ith realization of w.

Define $g(\theta; W)$ as:

$$g(\theta; W) = \frac{1}{n} \sum_{i=1}^{n} h(\theta; w_i),$$

where $W = \{w_n, w_{n-1}, \cdots, w_1\}$.

$g(\theta; W)$ is a $r \times 1$ vector for $r \geq k$.