

5 Time Series Analysis (時系列分析)

5.1 Introduction

代表的テキスト：

・ J.D. Hamilton (1994) *Time Series Analysis*

沖本・井上訳 (2006) 『時系列解析(上・下)』

・ A.C. Harvey (1981) *Time Series Models*

国友・山本訳 (1985) 『時系列モデル入門』

・ 沖本竜義 (2010) 『経済・ファイナンスデータの計量時系列分析』

1. Stationarity (定常性) :

Let y_1, y_2, \dots, y_T be time series data.

(a) Weak Stationarity (弱定常性) :

$$E(y_t) = \mu,$$

$$E((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \quad \tau = 0, 1, 2, \dots$$

The first and second moments do not depend on time.

The second moment depends on time difference, not time itself.

(b) Strong Stationarity (強定常性) :

Let $f(y_{t_1}, y_{t_2}, \dots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \dots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \dots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \dots, y_{t_r+\tau})$$

All the moments are same for all τ .

2. **Ergodicity** (エルゴード性) :

As time difference between two data is large, the two data become independent.

y_1, y_2, \dots, y_T is said to be ergodic in mean when \bar{y} converges in probability to $E(y_t)$.

3. **Auto-covariance Function** (自己共分散関数) :

$$E((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \quad \tau = 0, 1, 2, \dots$$

$$\gamma(\tau) = \gamma(-\tau)$$

4. **Auto-correlation Function** (自己相関関数) :

$$\rho(\tau) = \frac{E((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\text{Var}(y_t)} \sqrt{\text{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\text{Var}(y_t) = \text{Var}(y_{t-\tau}) = \gamma(0)$.

5. **Sample Mean** (標本平均) :

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^T y_t$$

6. **Sample Auto-covariance** (標本自己共分散) :

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^T (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

7. **Correlogram** (コレログラム, or 標本自己相関関数) :

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

8. **Lag Operator** (ラグ作要素) :

$$L^\tau y_t = y_{t-\tau}, \quad \tau = 1, 2, \dots$$

9. Likelihood Function (尤度関数) — Innovation Form :

The joint distribution of y_1, y_2, \dots, y_T is written as:

$$\begin{aligned} f(y_1, y_2, \dots, y_T) &= f(y_T|y_{T-1}, \dots, y_1)f(y_{T-1}, \dots, y_1) \\ &= f(y_T|y_{T-1}, \dots, y_1)f(y_{T-1}|y_{T-2}, \dots, y_1)f(y_{T-2}, \dots, y_1) \\ &\quad \vdots \\ &= f(y_T|y_{T-1}, \dots, y_1)f(y_{T-1}|y_{T-2}, \dots, y_1) \cdots f(y_2|y_1)f(y_1) \\ &= f(y_1) \prod_{t=2}^T f(y_t|y_{t-1}, \dots, y_1). \end{aligned}$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \dots, y_T) = \log f(y_1) + \sum_{t=2}^T \log f(y_t|y_{t-1}, \dots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \dots, y_1)$ is given by the normal distribution with conditional mean $E(y_t|y_{t-1}, \dots, y_1)$ and conditional variance $\text{Var}(y_t|y_{t-1}, \dots, y_1)$.

$\dots, y_1)$.

5.2 Autoregressive Model (自己回帰モデル or AR モデル)

1. AR(p) Model :

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L)y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p.$$

2. Stationarity (定常性) :

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one, y_t is stationary.

Suppose that the p solutions include imaginary numbers.

When the p solutions are outside unit circle, y_t is stationary.

3. **Partial Autocorrelation Coefficient** (偏自己相関係数), $\phi_{k,k}$:

The partial autocorrelation coefficient between y_t and y_{t-k} , denoted by $\phi_{k,k}$, is a measure of strength of the relationship between y_t and y_{t-k} , after removing influence of $y_{t-1}, \dots, y_{t-k+1}$.

$$\phi_{1,1} = \rho(1)$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{pmatrix}$$

⋮

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix}$$

Use Cramer's rule (クラメールの公式) to obtain $\phi_{k,k}$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

2. Rewriting the AR(1) model,

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \epsilon_t \\&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\&\quad \vdots \\&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.\end{aligned}$$

As s is large, ϕ_1^s approaches zero. \implies Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots$$

MA representation of AR model.

(MA will be discussed later.)

4. Mean of AR(1) process, μ

$$\begin{aligned}\mu &= E(y_t) = E(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \dots) \\ &= E(\epsilon_t) + \phi_1 E(\epsilon_{t-1}) + \phi_1^2 E(\epsilon_{t-2}) + \dots = 0\end{aligned}$$

5. Variance of AR(1) process, $\gamma(0)$

$$\begin{aligned}\gamma(0) &= V(y_t) = V(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \dots) \\ &= V(\epsilon_t) + V(\phi_1 \epsilon_{t-1}) + V(\phi_1^2 \epsilon_{t-2}) + \dots \\ &= V(\epsilon_t) + \phi_1^2 V(\epsilon_{t-1}) + \phi_1^4 V(\epsilon_{t-2}) + \dots \\ &= \sigma^2(1 + \phi_1^2 + \phi_1^4 + \dots) = \frac{\sigma^2}{1 - \phi_1^2}\end{aligned}$$

6. Autocovariance and autocorrelation functions of the AR(1) process:

Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \dots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}\gamma(\tau) &= E((y_t - \mu)(y_{t-\tau} - \mu)) = E(y_t y_{t-\tau}) \\ &= E\left((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \dots + \phi_1^{\tau-1} \epsilon_{t-\tau+1})y_{t-\tau}\right) \\ &= \phi_1^\tau E(y_{t-\tau} y_{t-\tau}) + E(\epsilon_t y_{t-\tau}) + \phi_1 E(\epsilon_{t-1} y_{t-\tau}) + \dots + \phi_1^{\tau-1} E(\epsilon_{t-\tau+1} y_{t-\tau}) \\ &= \phi_1^\tau \gamma(0) = \frac{\sigma^2 \phi_1^\tau}{1 - \phi_1^2}.\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^\tau.$$

7. Another Derivation of $\gamma(\tau)$:

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$E(y_t y_{t-\tau}) = \phi_1 E(y_{t-1} y_{t-\tau}) + E(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1\gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1\gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1\gamma(1) + \sigma^2 = \phi_1^2\gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1\gamma(0)$.

Autocovariance function $\gamma(\tau)$ is:

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) = \phi_1^2\gamma(\tau - 2) = \cdots = \phi_1^\tau\gamma(0).$$

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2}$$

8. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$
$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = 0$$

9. Estimation of AR(1) model:

(a) Likelihood function

$$\begin{aligned} \log f(y_T, \dots, y_1) &= \log f(y_1) + \sum_{t=2}^T \log f(y_t | y_{t-1}, \dots, y_1) \\ &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log\left(\frac{\sigma^2}{1 - \phi_1^2}\right) - \frac{1}{\sigma^2 / (1 - \phi_1^2)} y_1^2 \\ &\quad - \frac{T-1}{2} \log(2\pi) - \frac{T-1}{2} \log(\sigma^2) - \frac{1}{\sigma^2} \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2 \end{aligned}$$

$$\begin{aligned}
&= -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(\sigma^2) - \frac{1}{2} \log\left(\frac{1}{1 - \phi_1^2}\right) \\
&\quad - \frac{1}{2\sigma^2/(1 - \phi_1^2)} y_1^2 - \frac{1}{2\sigma^2} \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2
\end{aligned}$$

Note as follows:

$$\begin{aligned}
f(y_1) &= \frac{1}{\sqrt{2\pi\sigma^2/(1 - \phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1 - \phi_1^2)} y_1^2\right) \\
f(y_t|y_{t-1}, \dots, y_1) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (y_t - \phi_1 y_{t-1})^2\right)
\end{aligned}$$

$$\frac{\partial \log f(y_T, \dots, y_1)}{\partial \sigma^2} = -\frac{T}{2} \frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1 - \phi_1^2)} y_1^2 + \frac{1}{2\sigma^4} \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T, \dots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1 - \phi_1^2} + \frac{\phi_1}{\sigma^2} y_1^2 + \frac{1}{\sigma^2} \sum_{t=2}^T (y_t - \phi_1 y_{t-1}) y_{t-1} = 0$$

The MLE of ϕ_1 and σ^2 satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T} \left((1 - \tilde{\phi}_1^2) y_1^2 + \sum_{t=2}^T (y_t - \tilde{\phi}_1 y_{t-1})^2 \right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2 \tilde{\phi}_1}{1 - \tilde{\phi}_1^2} \right) / \sum_{t=2}^T y_{t-1}^2$$

(b) Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to ϕ_1 .

$$\hat{\phi}_1 = \frac{\sum_{t=2}^T y_{t-1} y_t}{\sum_{t=2}^T y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^T y_{t-1} \epsilon_t}{\sum_{t=2}^T y_{t-1}^2} = \phi_1 + \frac{(1/T) \sum_{t=2}^T y_{t-1} \epsilon_t}{(1/T) \sum_{t=2}^T y_{t-1}^2}$$

$$\rightarrow \phi_1 + \frac{E(y_{t-1} \epsilon_t)}{E(y_{t-1}^2)} = \phi_1$$

OLSE of ϕ_1 is a consistent estimator.

The following equations are utilized.

$$E(y_{t-1}\epsilon_t) = 0$$

$$E(y_{t-1}^2) = \text{Var}(y_{t-1}) = \gamma(0)$$

10. Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

Proof:

$y_{t-1}\epsilon_t$, $t = 1, 2, \dots, T$, are distributed with mean zero and variance $\frac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T) \sum_{t=1}^T y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1 - \phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t \longrightarrow N\left(0, \frac{\sigma_\epsilon^4}{1 - \phi_1^2}\right).$$

Next,

$$\frac{1}{T} \sum_{t=1}^T y_{t-1}^2 \longrightarrow E(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T}) \sum_{t=1}^T y_{t-1} \epsilon_t}{(1/T) \sum_{t=1}^T y_{t-1}^2} \longrightarrow N(0, 1 - \phi_1^2)$$

11. Some formulas:

(a) Central Limit Theorem

Random variables x_1, x_2, \dots, x_T are mutually independently distributed with mean μ and variance σ^2 .

Define $\bar{x} = (1/T) \sum_{t=1}^T x_t$.

Then,

$$\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} = \frac{\bar{x} - \mu}{\sigma / \sqrt{T}} \longrightarrow N(0, 1)$$

(b) Central Limit Theorem II

Random variables x_1, x_2, \dots, x_T are distributed with mean μ and variance σ^2 .

Define $\bar{x} = (1/T) \sum_{t=1}^T x_t$.

Then,

$$\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} \longrightarrow N(0, 1)$$

(c) Let x and y be random variables.

y converges in distribution to a distribution, and x converges in probability to a fixed value.

Then, xy converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \quad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$

12. **AR(1) + drift:** $y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

Mean:

Using the lag operator,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L$.

Multiply $\phi(L)^{-1}$ on both sides. Then, when $|\phi_1| < 1$, we have:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$\begin{aligned} E(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) \\ &= \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1} \end{aligned}$$

Example: AR(2) Model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$.

1. The stationarity condition is: two solutions of x from $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$ are outside the unit circle.
2. Rewriting the AR(2) model,

$$(1 - \phi_1 L - \phi_2 L^2)y_t = \epsilon_t.$$

Let $1/\alpha_1$ and $1/\alpha_2$ be the solutions of $\phi(x) = 0$.

Then, the AR(2) model is written as:

$$(1 - \alpha_1 L)(1 - \alpha_2 L)y_t = \epsilon_t,$$

which is rewritten as:

$$\begin{aligned} y_t &= \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \epsilon_t \\ &= \left(\frac{\alpha_1 / (\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2 / (\alpha_1 - \alpha_2)}{1 - \alpha_2 L} \right) \epsilon_t \end{aligned}$$

3. Mean of AR(2) Model:

When y_t is stationary, i.e., α_1 and α_2 are within the unit circle,

$$\mu = E(y_t) = E(\phi(L)\epsilon_t) = 0$$

4. Autocovariance Function of AR(2) Model:

$$\gamma(\tau) = E((y_t - \mu)(y_{t-\tau} - \mu)) = E(y_t y_{t-\tau})$$

$$\begin{aligned}
&= \mathbf{E}\left((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}\right) \\
&= \phi_1 \mathbf{E}(y_{t-1} y_{t-\tau}) + \phi_2 \mathbf{E}(y_{t-2} y_{t-\tau}) + \mathbf{E}(\epsilon_t y_{t-\tau}) \\
&= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}
\end{aligned}$$

The initial condition is obtained by solving the following three equations:

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_\epsilon^2,$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1),$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0).$$

Therefore, the initial conditions are given by:

$$\gamma(0) = \left(\frac{1 - \phi_2}{1 + \phi_2} \right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2},$$

$$\gamma(1) = \frac{\phi_1}{1 - \phi_2} \gamma(0) = \left(\frac{\phi_1}{1 - \phi_2} \right) \left(\frac{1 - \phi_2}{1 + \phi_2} \right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2}.$$

Given $\gamma(0)$ and $\gamma(1)$, we obtain $\gamma(\tau)$ as follows:

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \quad \text{for } \tau = 2, 3, \dots.$$

5. Another solution for $\gamma(0)$:

From $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2$,

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1\rho(1) - \phi_2\rho(2)}$$

where

$$\rho(1) = \frac{\phi_1}{1 - \phi_2}, \quad \rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2 + (1 - \phi_2)\phi_2}{1 - \phi_2}.$$

6. Autocorrelation Function of AR(2) Model:

Given $\rho(1)$ and $\rho(2)$,

$$\rho(\tau) = \phi_1\rho(\tau - 1) + \phi_2\rho(\tau - 2), \quad \text{for } \tau = 3, 4, \dots,$$

7. $\phi_{k,k}$ = Partial Autocorrelation Coefficient of AR(2) Process:

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix},$$

for $k = 1, 2, \dots$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Autocovariance Functions:

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1),$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0),$$