

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \quad \text{for } \tau = 3, 4, \dots$$

Autocorrelation Functions:

$$\rho(1) = \phi_1 + \phi_2\rho(1) = \frac{\phi_1}{1 - \phi_2},$$

$$\rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2,$$

$$\rho(\tau) = \phi_1\rho(\tau - 1) + \phi_2\rho(\tau - 2), \quad \text{for } \tau = 3, 4, \dots$$

$$\phi_{1,1} = \rho(1) = \frac{\phi_1}{1 - \phi_2}$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2$$

$$\phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}$$

$$= \frac{(\rho(3) - \rho(1)\rho(2)) - \rho(1)^2(\rho(3) - \rho(1)) + \rho(2)\rho(1)(\rho(2) - 1)}{(1 - \rho(1)^2) - \rho(1)^2(1 - \rho(2)) + \rho(2)(\rho(1)^2 - \rho(2))} = 0.$$

## 8. Log-Likelihood Function — Innovation Form:

$$\log f(y_T, \dots, y_1) = \log f(y_2, y_1) + \sum_{t=3}^T \log f(y_t | y_{t-1}, \dots, y_1)$$

where

$$f(y_2, y_1) = \frac{1}{2\pi} \begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}^{-1/2} \exp\left(-\frac{1}{2}(y_1 \ y_2) \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right),$$

$$f(y_t|y_{t-1}, \dots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2\right).$$

Note as follows:

$$\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} = \gamma(0) \begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} = \gamma(0) \begin{pmatrix} 1 & \phi_1/(1-\phi_2) \\ \phi_1/(1-\phi_2) & 1 \end{pmatrix}.$$

9. **AR(2) + drift:**  $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$

Mean:

Rewriting the AR(2)+drift model,

$$\phi(L)y_t = \mu + \epsilon_t$$

where  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$ .

Under the stationarity assumption, we can rewrite the AR(2)+drift model as follows:

$$y_t = \phi(L)^{-1} \mu + \phi(L)^{-1} \epsilon_t.$$

Therefore,

$$E(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) = \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1 - \phi_2}$$

**Example: AR( $p$ ) model:** Consider  $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$ .

### 1. Variance of AR( $p$ ) Process:

Under the stationarity condition (i.e., the  $p$  solutions of  $x$  from  $\phi(x) = 0$  are outside the unit circle),

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1 \rho(1) - \cdots - \phi_p \rho(p)}.$$

Note that  $\gamma(\tau) = \rho(\tau)\gamma(0)$ .

Solve the following simultaneous equations for  $\tau = 0, 1, \dots, p$ :

$$\gamma(\tau) = E((y_t - \mu)(y_{t-\tau} - \mu)) = E(y_t y_{t-\tau})$$

$$= \begin{cases} \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2) + \cdots + \phi_p\gamma(\tau - p), & \text{for } \tau \neq 0, \\ \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2) + \cdots + \phi_p\gamma(\tau - p) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}$$

## 2. Estimation of AR( $p$ ) Model:

### 1. OLS:

$$\min_{\phi_1, \dots, \phi_p} \sum_{t=p+1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p})^2$$

### 2. MLE:

$$\max_{\phi_1, \dots, \phi_p} \log f(y_T, \dots, y_1)$$

where

$$\log f(y_T, \dots, y_1) = \log f(y_p, \dots, y_2, y_1) + \sum_{t=p+1}^T \log f(y_t | y_{t-1}, \dots, y_1),$$

$$f(y_p, \dots, y_2, y_1) = (2\pi)^{-p/2} |V|^{-1/2} \exp \left( -\frac{1}{2} (y_1 \ y_2 \ \dots \ y_p) V^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \right)$$

$$V = \gamma(0) \begin{pmatrix} 1 & \rho(1) & \dots & \rho(p-2) & \rho(p-1) \\ \rho(1) & 1 & & \rho(p-3) & \rho(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \dots & \rho(1) & 1 \end{pmatrix}$$

$$f(y_t | y_{t-1}, \dots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp \left( -\frac{1}{2\sigma_\epsilon^2} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \dots - \phi_p y_{t-p})^2 \right)$$

### 3. Yule=Walker (ユール・ウォーカー) Equation:

Multiply  $y_{t-1}, y_{t-2}, \dots, y_{t-p}$  on both sides of  $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} +$

$\epsilon_t = y_t$ , take expectations for each case, and divide by the sample variance  $\hat{\gamma}(0)$ .

$$\begin{pmatrix} 1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-2) & \hat{\rho}(p-1) \\ \hat{\rho}(1) & 1 & & \hat{\rho}(p-3) & \hat{\rho}(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & \hat{\rho}(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p) \end{pmatrix}$$

where

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^T (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu}), \quad \hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}.$$

3. **AR(p) + drift:**  $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$

Mean:

$$\phi(L)y_t = \mu + \epsilon_t$$

where  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$ .

$$y_t = \phi(L)^{-1} \mu + \phi(L)^{-1} \epsilon_t$$

Taking the expectation on both sides,

$$\begin{aligned} E(y_t) &= \phi(L)^{-1} \mu + \phi(L)^{-1} E(\epsilon_t) = \phi(1)^{-1} \mu \\ &= \frac{\mu}{1 - \phi_1 - \phi_2 - \dots - \phi_p} \end{aligned}$$

#### 4. **Partial Autocorrelation of AR( $p$ ) Process:**

$\phi_{k,k} = 0$  for  $k = p + 1, p + 2, \dots$ .

## 5.3 MA Model

**MA (Moving Average, 移動平均) Model:**

### 1. MA( $q$ )

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1L + \theta_2L^2 + \cdots + \theta_qL^q.$$

## 2. Invertibility (反転可能性):

The  $q$  solutions of  $x$  from  $\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q = 0$  are outside the unit circle.

$\implies$  MA( $q$ ) model is rewritten as AR( $\infty$ ) model.

**Example: MA(1) Model:**  $y_t = \epsilon_t + \theta_1 \epsilon_{t-1}$

### 1. Mean of MA(1) Process:

$$E(y_t) = E(\epsilon_t + \theta_1 \epsilon_{t-1}) = E(\epsilon_t) + \theta_1 E(\epsilon_{t-1}) = 0$$

### 2. Autocovariance Function of MA(1) Process:

$$\begin{aligned}\gamma(0) &= E(y_t^2) = E(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = E(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2) \\ &= E(\epsilon_t^2) + 2\theta_1 E(\epsilon_t \epsilon_{t-1}) + \theta_1^2 E(\epsilon_{t-1}^2) = (1 + \theta_1^2) \sigma_\epsilon^2\end{aligned}$$

$$\gamma(1) = E(y_t y_{t-1}) = E((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-1} + \theta_1 \epsilon_{t-2})) = \theta_1 \sigma_\epsilon^2$$

$$\gamma(2) = E(y_t y_{t-2}) = E((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-2} + \theta_1 \epsilon_{t-3})) = 0$$

### 3. Autocorrelation Function of MA(1) Process:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \begin{cases} \frac{\theta_1}{1 + \theta_1^2}, & \text{for } \tau = 1, \\ 0, & \text{for } \tau = 2, 3, \dots \end{cases}$$

Let  $x$  be  $\rho(1)$ .

$$\frac{\theta_1}{1 + \theta_1^2} = x, \quad \text{i.e.,} \quad x\theta_1^2 - \theta_1 + x = 0.$$

$\theta_1$  should be a real number.

$$1 - 4x^2 > 0, \quad \text{i.e.,} \quad -\frac{1}{2} \leq \rho(1) \leq \frac{1}{2}.$$

#### 4. Invertibility Condition of MA(1) Process:

$$\begin{aligned}\epsilon_t &= -\theta_1\epsilon_{t-1} + y_t \\ &= (-\theta_1)^2\epsilon_{t-2} + y_t + (-\theta_1)y_{t-1} \\ &= (-\theta_1)^3\epsilon_{t-3} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2y_{t-2} \\ &\quad \vdots \\ &= (-\theta_1)^s\epsilon_{t-s} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2y_{t-2} + \cdots + (-\theta_1)^{t-s+1}y_{t-s+1}\end{aligned}$$

When  $(-\theta_1)^s\epsilon_{t-s} \rightarrow 0$ , the MA(1) model is written as the AR( $\infty$ ) model, i.e.,

$$y_t = -(-\theta_1)y_{t-1} - (-\theta_1)^2y_{t-2} - \cdots - (-\theta_1)^{t-s+1}y_{t-s+1} - \cdots + \epsilon_t$$

That is,  $|\theta_1| < 1$  represents the invertibility condition.

## 5. Partial Autocorrelation Function of MA(1) Process:

$$\phi_{1,1} = \rho(1) = \frac{\theta_1}{1 + \theta_1^2} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}$$

$$\Rightarrow \phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{-\rho(1)^2}{1 - \rho(1)^2} = \frac{-\theta_1^2}{1 + \theta_1^2 + \theta_1^4} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \end{pmatrix}$$

$$\Rightarrow \phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{vmatrix}} = \frac{\rho(1)^3}{1 - 2\rho(1)^2} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) & \rho(3) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(1) \\ \rho(3) & \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{4,1} \\ \phi_{4,2} \\ \phi_{4,3} \\ \phi_{4,4} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \\ \phi(4) \end{pmatrix}$$

$$\Rightarrow \phi_{4,4} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(2) & \rho(1) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(3) \\ \rho(3) & \rho(2) & \rho(1) & \rho(4) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) & \rho(3) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(1) \\ \rho(3) & \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & 0 & \rho(1) \\ \rho(1) & 1 & \rho(1) & 0 \\ 0 & \rho(1) & 1 & 0 \\ 0 & 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 & 0 \\ \rho(1) & 1 & \rho(1) & 0 \\ 0 & \rho(1) & 1 & \rho(1) \\ 0 & 0 & \rho(1) & 1 \end{vmatrix}} \neq 0$$

As a result,  $\phi_{k,k} \neq 0$  for all  $k = 1, 2, \dots$

## 6. Likelihood Function of MA(1) Process:

The autocovariance functions are:  $\gamma(0) = (1 + \theta_1^2)\sigma_\epsilon^2$ ,  $\gamma(1) = \theta_1\sigma_\epsilon^2$ , and  $\gamma(\tau) = 0$  for  $\tau = 2, 3, \dots$ .

The joint distribution of  $y_1, y_2, \dots, y_T$  is:

$$f(y_1, y_2, \dots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 & \theta_1 & 0 & \cdots & 0 \\ \theta_1 & 1 + \theta_1^2 & \theta_1 & \ddots & \vdots \\ 0 & \theta_1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 + \theta_1^2 & \theta_1 \\ 0 & \cdots & 0 & \theta_1 & 1 + \theta_1^2 \end{pmatrix}.$$

7. **MA(1) +drift:**  $y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}$

Mean of MA(1) Process:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where  $\theta(L) = 1 + \theta_1 L$ .

Taking the expectation,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$

**Example: MA(2) Model:**  $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

1. Autocovariance Function of MA(2) Process:

$$\gamma(\tau) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma_\epsilon^2, & \text{for } \tau = 0, \\ (\theta_1 + \theta_1\theta_2)\sigma_\epsilon^2, & \text{for } \tau = 1, \\ \theta_2\sigma_\epsilon^2, & \text{for } \tau = 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. let  $-1/\beta_1$  and  $-1/\beta_2$  be two solutions of  $x$  from  $\theta(x) = 0$ .

For invertibility condition, both  $\beta_1$  and  $\beta_2$  should be less than one in absolute value.

Then, the MA(2) model is represented as:

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$$

$$= (1 + \theta_1 L + \theta_2 L^2) \epsilon_t$$

$$= (1 + \beta_1 L)(1 + \beta_2 L) \epsilon_t$$

AR(∞) representation of the MA(2) model is given by:

$$\begin{aligned} \epsilon_t &= \frac{1}{(1 + \beta_1 L)(1 + \beta_2 L)} y_t \\ &= \left( \frac{\beta_1 / (\beta_1 - \beta_2)}{1 + \beta_1 L} + \frac{-\beta_2 / (\beta_1 - \beta_2)}{1 + \beta_2 L} \right) y_t \end{aligned}$$

### 3. Likelihood Function:

$$f(y_1, y_2, \dots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \theta_2 & & 0 \\ \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \ddots & \\ \theta_2 & \theta_1 + \theta_1\theta_2 & \ddots & \ddots & \theta_2 \\ & \ddots & \ddots & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 \\ 0 & & \theta_2 & \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 \end{pmatrix}$$

4. **MA(2) +drift:**  $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where  $\theta(L) = 1 + \theta_1L + \theta_2L^2$ .

Therefore,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu$$

**Example: MA( $q$ ) Model:**  $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

**1. Mean of MA( $q$ ) Process:**

$$E(y_t) = E(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}) = 0$$

**2. Autocovariance Function of MA( $q$ ) Process:**

$$\gamma(\tau) = \begin{cases} \sigma_\epsilon^2(\theta_0\theta_\tau + \theta_1\theta_{\tau+1} + \cdots + \theta_{q-\tau}\theta_q) = \sigma_\epsilon^2 \sum_{i=0}^{q-\tau} \theta_i\theta_{\tau+i}, & \tau = 1, 2, \dots, q, \\ 0, & \tau = q + 1, q + 2, \dots, \end{cases}$$

where  $\theta_0 = 1$ .

3. MA( $q$ ) process is stationary.

4. **MA( $q$ ) +drift:**  $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where  $\theta(L) = 1 + \theta_1L + \theta_2L^2 + \cdots + \theta_qL^q$ .

Therefore, we have:

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$