

# Econometrics II TA Session #06 \*

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# 1 Preliminary

Today, we review the random effect model and Hausman's specification test. An example of estimation is also explained in section 5.

- 2 Panel model: Basics
- 3 Random Effect Model
- 4.1 Hausman's Specification Error Test
- 4.2 Fixed Effect Model or Random Effect Model
- 5 Empirical Example
- 6 Appendix

## 2 Panel Model: Basics

Consider a dataset with time and individual specified structure. Then the model is described as follows:

$$y_{it} = X_{it}\beta + \nu_i + u_{it}, \quad (1)$$

where  $i$  represents individual ( $i = 1, 2, \dots, n$ ), and  $t$  stands for time ( $t = 1, 2, \dots, T$ ). In addition, we have the following assumptions:

- $\mathbb{E}[u_{it}] = 0$  for all  $i$ ;
- $\text{Var}[u_{it}] = \sigma_u^2$  for all  $i$  and  $t$ ;
- $\text{Cov}[u_{it}, u_{js}] = 0$  for  $i \neq j, t \neq s$ .

The variable  $\nu_i = z_i\alpha$  varies across the individual  $i$  and  $z_i$  may be observable (ex. race and sex) or unobservable (ex. skill and preference).

**Remark 1.1** The difference between fixed effect model and random effect model is whether  $z_i$  correlates with the dataset. Suppose that

- $z_i$  is correlated with  $X_{i,t}$  and  $\nu_i = z_i\alpha$  is constant term across  $i$ .  
→ fixed effect model
- $z_i$  is uncorrelated with  $X_{i,t}$ .  
→ random effect model

### 3 Random Effect Model

We can rewrite Eq. (1) in a stacked form with respect to time,  $t = 1, 2, \dots, T$  as follows:

$$y_i = X_i\beta + \epsilon_i, \quad \epsilon_i = 1_T\nu_i + u_i, \sim N(0, \Omega) \quad (2)$$

where  $y_i = (y_{i1}, y_{i2}, \dots, y_{iT})' \in \mathbb{R}^{T \times 1}$ ,  $X_i = (X'_{i1}, X'_{i2}, \dots, X'_{iT})' \in \mathbb{R}^{T \times k}$  and  $u_i = (u_{i1}, u_{i2}, \dots, u_{iT})' \in \mathbb{R}^{T \times 1}$ , respectively. Assumptions of this model are

- $\mathbb{E}[\nu_i|X] = \mathbb{E}[u_{it}|X] = 0$  for all  $i$ ;
- $\mathbb{E}[\nu_i|X] = \sigma_\nu^2$ ;
- $\text{Var}[u_{it}|X] = \sigma_u^2$ ;
- $\text{Cov}[\nu_i, \nu_j|X] = 0$  for  $i \neq j$ ;
- $\text{Cov}[u_{it}, u_{js}|X] = 0$  for  $i \neq j$  and  $t \neq s$ ;
- $\text{Cov}[\nu_i, u_{jt}|X] = 0$  for all  $i, j$  and  $t$ .

Here, the variance covariance matrix of  $\epsilon_i$  is given as follows:

$$\Omega := \mathbb{E}[\epsilon_i\epsilon_i'|X] = \begin{pmatrix} \sigma_u^2 + \sigma_\nu^2 & \sigma_\nu^2 & \dots & \sigma_\nu^2 \\ \sigma_\nu^2 & \sigma_u^2 + \sigma_\nu^2 & \dots & \vdots \\ \vdots & \dots & \ddots & \vdots \\ \sigma_\nu^2 & \dots & \dots & \sigma_u^2 + \sigma_\nu^2 \end{pmatrix}.$$

Again, in a matrix form with respect to  $i$ , we have

$$y = X\beta + \nu + u, \quad \nu + u \sim N(0, I_n \otimes \Omega), \quad (3)$$

where  $y = (y'_1, y'_2, \dots, y'_n)' \in \mathbb{R}^{nT \times 1}$ ,  $X = (X'_1, X'_2, \dots, X'_n)' \in \mathbb{R}^{nT \times k}$  and  $u = (u'_1, u'_2, \dots, u'_n)' \in \mathbb{R}^{nT \times 1}$ . The likelihood function of Eq. (3) is given by

$$L(\beta, \sigma_u^2, \sigma_\nu^2) = (2\pi)^{-nT/2} |I_n \otimes \Omega|^{-1/2} \times \exp \left[ -\frac{1}{2} (y - X\beta)' (I_n \otimes \Omega)^{-1} (y - X\beta) \right].$$

The **maximum likelihood estimator (MLE)** of this model becomes

$$\hat{\beta}_{re} = (X'(I_n \otimes \Omega)X)^{-1} X'(I_n \otimes \Omega)y.$$

This is equivalent to the **generalized least squares (GLS)** estimator, which has the **efficiency**.

## 4 Hausman Test and Model Selection

The following discussion refers to chapter 11 of Greene (2012). Because the key consideration in choosing a random effect model and fixed effect model is whether  $v_i$  and  $X_{it}$  are correlated, it is important to have a method for testing this assumption. The specification test devised by Hausman (1978) is used for testing the orthogonality of the individual effects and the regressors.

### 4.1 Hausman's Specification Error Test

Suppose that we have the following standard regression model:

$$y = X\beta + u, \quad u \sim N(0, \sigma^2 I_n).$$

Here, assume that we check the orthogonality condition of the regressors. The null and alternative hypotheses are given by

$H_0 : \text{Cov}[X, u] = \mathbb{E}[X'u] = 0$  ( $X$  is not correlated with  $u$ );

$H_1 : \mathbb{E}[X'u] \neq 0$  ( $X$  is correlated with  $u$ ).

Suppose that we have two estimators  $\hat{\beta}_0$  and  $\hat{\beta}_1$ , which have the following properties.

	$H_0$	$H_1$
$\hat{\beta}_0$	○ consistency/○ efficiency	× consistency
$\hat{\beta}_1$	○ consistency/× efficiency	○ consistency

For instance,  $\hat{\beta}_0$  is OLSE, while  $\hat{\beta}_1$  is IV estimator. Therefore, under  $H_0$ , the estimates should not differ systematically, and a test can be based on the difference. The essential ingredient for this test is

$$\text{Var}[\hat{\beta}_1 - \hat{\beta}_0] = \text{Var}[\hat{\beta}_1] + \text{Var}[\hat{\beta}_0] - \text{Cov}[\hat{\beta}_1, \hat{\beta}_0] - \text{Cov}[\hat{\beta}_0, \hat{\beta}_1]. \quad (4)$$

In the above relationship, we can show that  $\text{Cov}[\hat{\beta}_0, \hat{\beta}_1] = \text{Var}[\hat{\beta}_0]$  under some conditions (and explain how to prove in the Appendix). Therefore, we have the following test statistic which is based on the **Wald criterion**:

$$(\hat{\beta}_1 - \hat{\beta}_0)' [\text{Var}(\hat{\beta}_1) - \text{Var}(\hat{\beta}_0)]^{-1} (\hat{\beta}_1 - \hat{\beta}_0) \rightarrow \chi^2(k).$$

## 4.2 Fixed Effect Model or Random Effect Model

Before we learn the case of the panel data model, recall the assumptions of the random effect model and the fixed effect model (Remark1). We set  $\hat{\beta}_{fe}$  : the estimator of FE model and  $\hat{\beta}_{re}$  : the estimator of RE model. Asymptotic properties of these estimators are shown as below:

$$\begin{aligned}\sqrt{T}(\hat{\beta}_{re} - \beta_{re}) &\rightarrow N(0, V_0); \\ \sqrt{T}(\hat{\beta}_{fe} - \beta_{fe}) &\rightarrow N(0, V_1),\end{aligned}$$

where  $V_0$  is the **Cramer-Rao lower bound** ( $T$  is the sample size).

Under  $H_0$ , we have

$$\text{Var}[\hat{\beta}_{fe} - \hat{\beta}_{re}] = \text{Var}[\hat{\beta}_{fe}] - \text{Var}[\hat{\beta}_{re}].$$

Test statistic is given by

$$(\hat{\beta}_{fe} - \hat{\beta}_{re})' \left[ \text{Var}[\hat{\beta}_{fe}] - \text{Var}[\hat{\beta}_{re}] \right]^{-1} (\hat{\beta}_{fe} - \hat{\beta}_{re}) \rightarrow \chi^2(k). \quad (5)$$

## 5 Empirical Example

Today, we analyze a panel data model of the US states production data, contained in plm package. We estimate the following model:

$$\log(gsp)_{it} = \log(pcap)_{it}\beta_1 + \log(pc)_{it}\beta_2 + \log(emp)_{it}\beta_3 + unemp_{it} + \nu_i + u_{it}.$$

The assumptions on this model is same as Section1.

- $gsp_{it}$ : gross state product
- $pcap_{it}$ : public capital stock
- $pc_{it}$ : private capital stock
- $emp_{it}$ : labor input measured by the employment in non agricultural payrolls
- $unemp_{it}$ : state unemployment rate

The R code is given as follows.

```
library(plm)
library(stargazer)
data("Produc", package = "plm")

panel_est_random<- plm(log(gsp) ~ log(pcap) + log(pc) + log(emp)
+ unemp, data = Produc, index = c("state","year"),model="random")
panel_est_fixed<- plm(log(gsp) ~ log(pcap) + log(pc) + log(emp)
+ unemp, data = Produc, index = c("state","year"),model="within")

phtest(panel_est_random,panel_est_fixed)
#H_0: X and u are independent. (Use random effect model.)
```

```
#H_1: X and u are NOT independent. (Use fixed effect model.)
#p=0.049<0.05, H_0 is rejected: use fixed effect model.
stargazer(panel_est_fixed)
```

We can test which model is appropriate to estimate by `phptest` function. As a result,  $H_0$  is rejected and we should conduct the estimate with a fixed effect model. The result is given as Table1.

Table1

<i>Dependent variable:</i>	
	log(gsp)
log(pcap)	-0.026 (0.029)
log(pc)	0.292*** (0.025)
log(emp)	0.768*** (0.030)
unemp	-0.005*** (0.001)
Observations	816
R <sup>2</sup>	0.941
Adjusted R <sup>2</sup>	0.937
F Statistic	3,064.808*** (df = 4; 764)
<i>Note:</i>	*p<0.1; **p<0.05; ***p<0.01

## 6 Appendix

Now we get to show that  $\text{Cov}[\hat{\beta}_0, \hat{\beta}_1] = \text{Var}[\hat{\beta}_0]$  under some conditions. In Hausman (1978), more detail of the specification test is explained.

**Lemma 6.1** Consider two estimators  $\hat{\beta}_0, \hat{\beta}_1$  which are both consistent and asymptotically normally distributed with  $\hat{\beta}_0$  attaining the asymptotic Cramer-Rao bound so  $\sqrt{T}(\hat{\beta}_0 - \beta_0) \rightarrow N(0, V_0)$  and  $\sqrt{T}(\hat{\beta}_1 - \beta_1) \rightarrow N(0, V_1)$ , where  $V_0$  is the inverse of Fisher's Information matrix. Let  $\hat{q} = \hat{\beta}_1 - \hat{\beta}_0$ . Then, the limiting distribution of  $\sqrt{T}(\hat{\beta}_0 - \beta)$  and  $\sqrt{T}\hat{q}$  have zero covariance, that is,  $\text{Cov}[\hat{\beta}_0, \hat{\beta}_1 - \hat{\beta}_0] = 0$ .

*Proof.* Assume that we define

$$C \equiv \text{Cov}[\hat{\beta}_0, \hat{\beta}_1 - \hat{\beta}_0] = 0, \quad (6)$$

to prove the above lemma. Suppose  $\hat{\beta}_0$  and  $\hat{q}$  are not orthogonal. Since  $\hat{q}$  converges to zero in probability, we can define a new estimator:

$$\hat{\beta}_2 = \hat{\beta}_0 + rA\hat{q}, \quad (7)$$

where  $r$  is a scalar and  $A$  is an arbitrary matrix to be chosen. The new estimator is consistent and asymptotically normal with asymptotic variance:

$$\text{Var}[\hat{\beta}_2] = \text{Var}[\hat{\beta}_0] + rAC + rC'A' + r^2A(\text{Var}[\hat{q}])A'.$$

Difference between  $\text{Var}[\hat{\beta}_2]$  and  $\text{Var}[\hat{\beta}_0]$  is

$$F(r) = \text{Var}[\hat{\beta}_2] - \text{Var}[\hat{\beta}_0] = rAC + rC'A' + r^2A(\text{Var}[\hat{q}])A'.$$

By taking derivatives with respect to  $r$  and choosing  $A = -C'$ , we have

$$F'(r) = -2C'C + 2rC'(\text{Var}[\hat{q}])C,$$

in the case of  $r = 0$ ,  $F(0) = 0$  and  $F'(0) = -2C'C \leq 0$  (negative semi-definite). From this result,  $F(r)$  is negative definite for small  $r$ . Thus, we can find that there is a contradiction unless  $C'C = 0$  ( $C = 0$ ) since  $\hat{\beta}_0$  is asymptotically efficient implies  $F(r) \geq 0$ .  $\square$

## References

- [1] William, H., Greene (2012) "*Econometric analysis Seventh Edition*", Pearson.
- [2] J. A. Hausman (1978) "*SPECIFICATION TESTS IN ECONOMETRICS*", *Econometrica*, Vol. 46, No.6, 1251-1271.