Econometrics II TA Session #08 (Enhanced and Corrected Version) *

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1 Preliminary

Today, we review the (linear) generalized method of moments (GMM). The empirical example refers to the paper of monetary economics.

2 Model Setting and Order Condition

The following discussion is explained in Chapter 3 of Hayashi (2001) and Chapter 8 of Wooldridge (2010). Suppose the following regression model:

$$y_i = x_i\beta + u_i \ (i = 1, 2, \cdots, n), \tag{1}$$

where $x_i = (1, x_{i_2}, \ldots, x_{i_K}) \in \mathbb{R}^{1\times K}$ and $\beta \in \mathbb{R}^{K\times 1}$. Assume that the instrumental variables in Eq. (1) are $z_{i1}, \ldots, z_{iM}$. In addition, the endogeneous variable is x_{iK}. Then, we have $1 \times L$ vector of the exogeneous variables $z_i = (1, x_{i2}, \cdots, x_{i(K-1)}, z_{i1}, \cdots, z_{iM})$. Assume that the orthogonality condition of the regressors and error terms are violated. Then, we must estimate Eq. (1) by using the external instrumental variables and we have

$$\mathbb{E}[z_i'(y_i - x_i\beta)] = 0. \tag{2}$$

Thus, multiplying both sides of Eq. (1) by z_i' and taking the expectations yields

$$\mathbb{E}[z_i'x_i]\beta = \mathbb{E}[z_i'y_i]$$

To estimate Eq. (1) by the above relationship, **the rank condition** is important.

Assumption 2.1 (Rank Condition) The matrix $\mathbb{E}[z_i'x_i] \in \mathbb{R}^{L\times K}$ is full column rank.

The above assumption is a sufficient condition to derive a solution to the simultaneous equation system. Since $\mathbb{E}[z_i'x_i] \in \mathbb{R}^{L\times K}$, **(Assumption 1.1)** requires the columns of this matrix to be lineary independent. A necessary condition for the rank condition is **the order condition**, explained as follows[*1].

[*1] The model setting used in this class is the same as the Econometrics I TA session #14 (2SLS). Therefore, $L(= K + M)$ is larger than K. However, in general case, we may have z_i as a $1 \times K$ vector if we can identify the linear model.

Remark 1.2 (Order Condition for Identification) A necessary condition for identification is L (#exogeneous variables) $\geq K$ (#regressors).

- $L > K$: Over Identified ($\rightarrow$ 2SLS or **GMM**)
- $L = K$: Just Identified ($\rightarrow$ IV)
- $L < K$: Under Identified (*We can NOT estimate the regression model.)

3 GMM Estimator

If Eq. (1) is **just identified**, we can derive the **IV estimator** as

$$\hat{\beta}_{IV} = \left(\frac{1}{n} \sum_{i=1}^{n} z_i' x_i \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} z_i' y_i \right).$$

However, in the case that the model is over identified, we cannot generally choose $K \times 1$ vector $\hat{\beta}$ to satisfy L equations. Therefore, we must use other methodologies like 2SLS or GMM.

3.1 Deriving the GMM Estimator

In the case of the **over identified** model, we choose $\hat{\beta}$ so that $\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)$ is as close to zero as possible. Generally, we use a weighting matrix in a quadratic form:

$$\hat{\beta}_{GMM} = \arg\min_{\beta} \left[\left(\frac{1}{n} \sum_{i=1}^{n} z_i'(y_i - x_i\beta) \right)' W \left(\frac{1}{n} \sum_{i=1}^{n} z_i'(y_i - x_i\beta) \right) \right]. \tag{3}$$

In practice, we solve the following minimization problem to derive the estimator of Eq. (1):

$$\min_{\beta} \; (Z'(y - X\beta))' \, W \, (Z'(y - X\beta)). \tag{4}$$

Then, using multivariate calculus, we have a closed form solution:

$$\hat{\beta}_{GMM} = (X'ZWZ'X)^{-1}(X'ZWZ'y). \tag{5}$$

Surely, we must estimate W in Eq. (5). Therefore, we represent $\hat{\beta}_{GMM}$ as $\hat{\beta}_{GMM} = (X'Z\hat{W}Z'X)^{-1}(X'Z\hat{W}Z'y)$.

Assumption 3.1 Assume that $\hat{W} \xrightarrow{p} W$ as $n \to \infty$, where W is a random, symmetric, $L \times L$ positive definite matrix.

In this class, we regard the inverse matrix of the variance–covariance matrix of $Z'u$ as W. Suppose that $\mathrm{Var}(u) = \sigma^2\Omega$. Then, $\mathrm{Var}(Z'u) = \mathbb{E}(Z'u(Z'u)') = \sigma^2 Z'\Omega Z = W^{-1}$.

3.2 Properties of the Linear GMM Estimator

By the previous subsection, the solution of Eq. (3) is obtained as

$$\hat{\beta}_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'y. \tag{6}$$

This estimator has **consistency** and **asymptotic normality**.

Theorem 3.2 Under the **(Assumption 2.1)** , **(Assumption 1.2)**, **(Assumption 3.1)** and the orthogonality condition, $\hat{\beta}_{GMM}$ has **consistency and asymptotic normality**.

Proof. We begin with the proof of the consistency. By Eq. (6), we have

$$\begin{aligned}\hat{\beta}_{GMM} &= (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'(X\beta + u)\\ &= \beta + (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u\\ &= \beta + \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\\ &\quad\times\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'u\right).\end{aligned} \tag{7}$$

Here, we assume that $\frac{1}{n}X'Z \xrightarrow{p} M_{xz}$ and $\frac{1}{n}Z'\Omega Z \xrightarrow{p} M_{z\Omega z}$. From the orthogonality condition, we can prove that the GMM estimator has consistency. Next, we prove the asymptotic normality. From the CLT, we can say that

$$\frac{1}{\sqrt{n}}Z'u \xrightarrow{d} N(0, \sigma^2 M_{z\Omega z}).$$

From the above relationship,

$$\begin{aligned}\sqrt{n}(\hat{\beta}_{GMM} - \beta) &= \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\\ &\quad\times\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right).\end{aligned}$$

By using this result, $\hat{\beta}_{GMM}$ is asymptotically distributed as:

$$\sqrt{n}(\hat{\beta}_{GMM} - \beta) \xrightarrow{d} N(0, \sigma^2(M_{xz}{M_{z\Omega z}}^{-1}M'_{xz})^{-1}). \tag{8}$$

Practically, we use $s^2 := \frac{1}{n-K}(y - X\hat{\beta}_{GMM})'\Omega^{-1}(y - X\hat{\beta}_{GMM})$ instead of σ^2.

□

4 Empirical Example

In this class, we would like to introduce a paper that applies the linear GMM estimation. L'œillent and Licheron (2012) estimates an extended Taylor rule to clarify the sensitivity of the European Central Bank (ECB) to oil price fluctuations. The original Taylor rule is refered to describe the target of the interest rate set by the central bank:

$$i_t^* = \bar{i}_t + \beta(\pi_t - \pi^*) + \gamma(y_t - y^*),$$

where $\bar{i}_t$ is the equilibrium nominal interest rate, π_t is the inflation rate, y_t is the output growth rate. π^* and y^* are targets of production level and interest rate. They added a smoothing parameter ρ and 12-month variation of nominal oil prices Δo_t, which indicates whether the oil prices increase or not. Therefore, the above equation is rewritten as

$$i_t = \rho i_{t-1} + (1-\rho)[\bar{i}_t + \beta(\pi_t - \pi^*) + \gamma(y_t - y^*) + \lambda \Delta o_t] + u_t.$$

In practice, their modified Taylor rule is described as follows:

$$i_t = \alpha_1 + \alpha_2 i_{t-1} + \alpha_3(\pi_t^e - \pi^*) + \alpha_4(y_t - y^*)^e + \alpha_5 \Delta o_t + e_t. \tag{9}$$

Remind that $(\pi_t^e - \pi^*)$ implies the difference between the consumption index calculated by the ECB and the inflation target (2%). Besides, they derived the monthly forecast real GDP by the Hodrick-Prescott filter. Therefore, $(y_t - y^*)^e$ is the deviation of the interpolated GDP from the trend. They estimated by the two-step efficient GMM estimator initiated by Hansen (1982). The instruments are all explanatory variables and ex-post inflation.

Table 1
Estimation results.

	[1]	[2]	[3]	[4]	[5]
Constant	**3.357*****	**0.144*****	0.081	0.082	0.067
	(0.089)	(0.051)	(0.049)	(0.050)	(0.067)
i_{t-1}		**0.960*****	**0.975*****	**0.964*****	**0.976*****
		(0.016)	(0.015)	(0.018)	(0.020)
$(\pi_t - \pi^*)$	**1.956*****	0.107	**0.121***	**0.176****	**0.142***
	(0.291)	(0.074)	(0.064)	(0.081)	(0.080)
$(y_t - y_t^*)$	**0.383***	**0.235*****	**0.195*****	**0.211*****	**0.203*****
	(0.220)	(0.042)	(0.037)	(0.040)	(0.055)
Δo_t			**0.0012*****		
			(0.0003)		
Δo_t^+				**0.0022****	
				(0.001)	
Δo_t^-				−0.0025	
				(0.003)	
NOPI					**0.0211****
					(0.009)
NOPD					0.0067
					(0.009)
Implied coefficients					
ρ	–	**0.960**	**0.975**	**0.964**	**0.976**
β	**1.956**	2.675	**4.840**	**4.889**	**5.917**
γ	**0.383**	**5.875**	**7.800**	**5.861**	**8.458**
λ	–	–	**0.048**	–	–
λ^+	–	–	–	**0.061**	**0.879**
λ^-	–	–	–	−0.069	0.279
Observations	117	118	118	118	118
Adjusted R2	0.117	0.980	0.982	0.979	0.976
Hansen *J*-test	20.325	3.093	7.082	5.658	5.282
P-value	[0.000]	[0.378]	[0.132]	[0.130]	[0.152]

- The first two columns: comparing the outcome of the standard rule with and without smoothing parameter
- The third column: estimating Eq. (9)
- The fourth column: checking the assumption of an asymmetric behavior of the ECB
- The last column: assessing a potential nonlinear reaction of the oil prices

References

[1] L.P. Hansen (1982) *"Large sample properties of generalized method of moments estimators"*, Econometrica 50, 1029-1054.

[2] Fumio, Hayashi (2000) *"ECONOMETRICS"*, Princeton University Press.

[3] G. L'œillent and J. Licheron (2012) *"How does the European Central Bank react to oil prices?"*, Economic Letters 116, 445-447.

[4] J. M. Wooldridge (2010) *"Econometric Analysis of Cross Section and Panel Data (2nd Eddition)"*, The MIT Press.