Econometrics II TA Session #13*

Kenta KUDO †

January 28th, 2019

Contents

* All comments welcome!

† E-mail: vge011kk@student.econ.osaka-u.ac.jp

1 Solutions to Excercise 2—Question 1

Question 1

(1)

At first, consider the stacked regression model:

$$y = X\beta + u,$$

where $y = (y_1, y_2, \cdots, y_n)'$, $u = (u_1, u_2, \cdots, u_n)'$ and the data matrix X is

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}.$$

The OLS estimator of the above model is

$$\begin{aligned} \hat{\beta} &= b + (X'X)^{-1}X'u \\ &= b + \left(\frac{1}{n}\sum_{i=1}^{n} X_i'X_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i'u_i\right). \end{aligned} \tag{1}$$

By taking the probability limit on both sides, we have

$$\begin{aligned} \operatorname*{plim}_{n\to\infty} \hat{\beta} &= \operatorname*{plim}_{n\to\infty} \left[b + \left(\frac{1}{n}\sum_{i=1}^{n} X_i'X_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i'u_i\right)\right] \\ &= b + \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} X_i'X_i\right)^{-1} \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} X_i'u_i\right). \end{aligned} \tag{2}$$

Here, suppose that $\mathbb{E}[X_i'X_i] < \infty$. By the WLLN, we have

$$\begin{aligned} \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} X_i'X_i\right) &\to \mathbb{E}[X_i'X_i] < \infty \\ \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} X_i'u_i\right) &\to \mathbb{E}[X_i'u_i] \neq 0. \end{aligned}$$

In the Eq. (2), the orthogonality condition is violated, so we can say

$$\operatorname*{plim}_{n\to\infty} \hat{\beta} \nrightarrow b.$$

(2)

Because this model is over identified, we have to estimate the regressor by GMM. The estimator satisfies the following relationship:

$$\hat{\beta}_{GMM} = \arg\min_{\beta} \left[\left(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\right)' W \left(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\right)\right]. \tag{3}$$

In practice, we solve the following minimization problem to derive the GMM estimator of Eq. (1):

$$\min_{\beta} \; (Z'(y - X\beta))' W (Z'(y - X\beta)). \tag{4}$$

The weight matrix W is a random, symmetric, $L \times L$ positive definite matrix. We use the estimator of W, $\hat{W}$, instead of the weight matrix. Assume that $\hat{W} \xrightarrow{p} W$. When we estimate W, we regard the inverse matrix of the variance–covariance matrix of $Z'u$ as W. Suppose that $\mathrm{Var}(u) = \sigma^2\Omega$. Then, $\mathrm{Var}(Z'u) = \mathbb{E}(Z'u(Z'u)') = \sigma^2 Z'\Omega Z = W^{-1}$. The GMM estimator is

$$\hat{\beta}_{GMM} = (X'Z\hat{W}Z'X)^{-1}(X'Z\hat{W}Z'y). \tag{5}$$

(3)

The solution of Eq. (3) is obtained as

$$\hat{\beta}_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'y. \tag{6}$$

We begin with the proof of **the consistency**.

$$\begin{aligned}\hat{\beta}_{GMM} &= (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'(X\beta + u)\\ &= \beta + (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u\\ &= \beta + \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\\ &\quad \times \left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'u\right). \end{aligned} \tag{7}$$

Assume that $\frac{1}{n}X'Z \xrightarrow{p} M_{xz}$ and $\frac{1}{n}Z'\Omega Z \xrightarrow{p} M_{z\Omega z}$. From the orthogonality condition, we can prove that the GMM estimator has consistency.

Next, we show that the GMM estimator has **the asymptotic normality**. Then, from the CLT, we have

$$\frac{1}{\sqrt{n}}Z'u \xrightarrow{d} N(0, \sigma^2 M_{z\Omega z}).$$

By rewriting the Eq. (7), we can establish the following relationship:

$$\begin{aligned}\sqrt{n}(\hat{\beta}_{GMM} - \beta) &= \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\\ &\quad \times \left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right).\end{aligned}$$

By using this result, $\hat{\beta}_{GMM}$ is asymptotically distributed as:

$$\sqrt{n}(\hat{\beta}_{GMM} - \beta) \xrightarrow{d} N(0, \sigma^2(M_{xz}{M_{z\Omega z}}^{-1}M'_{xz})^{-1}). \tag{8}$$

2 Solutions to Excercise 2—Question 2

Question 2

(12)

Under H_0, $\hat{\beta}$ is consistent and efficient. Thus we should use $\hat{\beta}$. Note that $\tilde{\beta}$ is also consistent and not efficient.

(13)

Under H_1, only $\tilde{\beta}$ is consistent. Thus we should use $\tilde{\beta}$.