6.2.3 誤差項に系列相関がある場合

回帰モデル

$$Y_i = \alpha + \beta X_i + u_i$$

$$u_i = \rho u_{i-1} + \epsilon_i \qquad\qquad i = 2, 3, \cdots, n$$

$\epsilon_2, \epsilon_3, \cdots, \epsilon_n$ は互いに独立で，すべての i について $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$ を仮定する。

u_i を消去すると，

$$(Y_i - \alpha - \beta X_i) = \rho(Y_{i-1} - \alpha - \beta X_{i-1}) + \epsilon_i$$

または

$$(Y_i - \rho Y_{i-1}) = \alpha(1 - \rho) + \beta(X_i - \rho X_{i-1}) + \epsilon_i$$

と書き直すことが出来る。

$\theta = (\alpha, \beta, \sigma^2, \rho)$ とする。

$$\log f(Y_i;\theta) = -\frac{1}{2}\log(2\pi\sigma^2) \\ -\frac{1}{2\sigma^2}\Big((Y_i - \rho Y_{i-1}) - \alpha(1-\rho) - \beta(X_i - \rho X_{i-1})\Big)^2$$

尤度関数は,

$$\log l(\theta) = \sum_{i=2}^{n}\log f(Y_i;\theta) = -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\sigma^2) \\ -\frac{1}{2\sigma^2}\sum_{i=2}^{n}\Big((Y_i - \rho Y_{i-1}) - \alpha(1-\rho) - \beta(X_i - \rho X_{i-1})\Big)^2$$

となる。

尤度関数をそれぞれ α，β，σ^2，ρ について微分し，ゼロとおく。

$$\frac{\partial \log l(\theta)}{\partial \alpha} = \frac{1-\rho}{\sigma^2}\sum_{i=2}^{n}\Big((Y_i - \rho Y_{i-1}) \\ -\alpha(1-\rho) - \beta(X_i - \rho X_{i-1})\Big) = 0$$

$$\frac{\partial \log l(\theta)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=2}^{n}(X_i - \rho X_{i-1})\Big((Y_i - \rho Y_{i-1}) \\ -\alpha(1-\rho) - \beta(X_i - \rho X_{i-1})\Big) = 0$$

$$\frac{\partial \log l(\theta)}{\partial \sigma^2} = -\frac{n-1}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=2}^{n}\Big((Y_i - \rho Y_{i-1}) \\ -\alpha(1-\rho) - \beta(X_i - \rho X_{i-1})\Big)^2 = 0$$

$$\frac{\partial \log l(\theta)}{\partial \rho} = \frac{1}{\sigma^2}\sum_{i=2}^{n}(Y_{i-1} - \alpha - \beta X_{i-1})\Big((Y_i - \alpha - \beta X_i) \\ -\rho(Y_{i-1} - \alpha - \beta X_{i-1})\Big) = 0$$

$(Y_i - \alpha - \beta X_i) - \rho(Y_{i-1} - \alpha - \beta X_{i-1})$ は

$(Y_i - \rho Y_{i-1}) - \alpha(1-\rho) - \beta(X_i - \rho X_{i-1})$ を書き直したもの。

4 つの連立方程式を解いて，最尤推定量 $\widehat{\alpha}$，$\widehat{\beta}$，$\widehat{\sigma}^2$，$\widehat{\rho}$ が得られる。

$\longrightarrow$ 下記のように収束計算によって求める。

(i) 初期段階では，$\widehat{\rho} = 0$ とする。

(ii) $X_i^* = X_i - \widehat{\rho} X_{i-1}$

$Y_i^* = Y_i - \widehat{\rho} Y_{i-1}$

(iii) $\begin{pmatrix} \widetilde{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{pmatrix} n-1 & \sum_{i=2}^n X_i^* \\ \sum_{i=2}^n X_i^* & \sum_{i=2}^n X_i^{*2} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=2}^n Y_i^* \\ \sum_{i=2}^n X_i^* Y_i^* \end{pmatrix}$

(iv) $\widehat{\alpha} = \dfrac{\widetilde{\alpha}}{1 - \widehat{\rho}}$

(v) $\widehat{u}_i = Y_i - \widehat{\alpha} - \widehat{\beta} X_i$

(vi) $\widehat{\sigma}^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=2}^n (\widehat{u}_i - \widehat{\rho}\widehat{u}_{i-1})^2$

(vii) $\widehat{\rho} = \dfrac{\sum_{i=2}^n \widehat{u}_i \widehat{u}_{i-1}}{\sum_{i=2}^n \widehat{u}_{i-1}^2}$

(viii) ステップ (ii) ～ (vii) を，収束するまで繰り返し計算する。

6.3 尤度比検定

n 個の確率変数 $X_1, X_2, \cdots, X_n$ は互いに独立で，同じ確率分布 $f(x) \equiv f(x;\theta)$ とする。

尤度関数は,

$$l(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

となる。

θ の制約つき最尤推定量を $\widetilde{\theta}$，制約無し最尤推定量を $\widehat{\theta}$ とする。

制約の数を G 個とする。

$\dfrac{l(\widetilde{\theta})}{l(\widehat{\theta})}$ を尤度比と呼ぶ

検定方法 1：　尤度比がある値より小さいときに，帰無仮説を棄却する。すなわち，

$$\frac{l(\widetilde{\theta})}{l(\widehat{\theta})} < c$$

となるときに，帰無仮説を棄却する。この場合，c を次のようにして求める必要がある。

$$\int \cdots \int \prod_{i=1}^{n} f(x_i; \widetilde{\theta}) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \alpha$$

ただし，α は有意水準（帰無仮説が正しいときに，帰無仮説を棄却する確率）を表す。

検定方法 2（大標本検定）：　または，$n \longrightarrow \infty$ のとき，

$$-2 \log \frac{l(\widetilde{\theta})}{l(\widehat{\theta})} \longrightarrow \chi^2(G)$$

となる。

この検定を尤度比検定と呼ぶ。

例 1： 正規母集団 $N(\mu, \sigma^2)$ からの標本値 $x_1, x_2, \cdots, x_n$ を用いて，σ^2 が既知のとき，帰無仮説 H_0：$\mu = \mu_0$，H_1：$\mu \neq \mu_0$ の尤度比検定を行う。

σ^2 が既知のとき，尤度関数 $l(\mu)$ は，

$$l(\mu) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\Bigl(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\Bigr)$$

となる。

$l(\mu)$ を最大にする μ と $\log l(\mu)$ を最大にする μ は同じになる。

μ の最尤推定量は,

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \equiv \overline{X}$$

となる。

尤度比検定統計量は,

$$\frac{l(\mu_0)}{l(\overline{X})} = \frac{\exp\Bigl(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\Bigr)}{\exp\Bigl(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2\Bigr)}$$

$$= \exp\Bigl(-\frac{1}{2\sigma^2/n}(\overline{X}-\mu_0)^2\Bigr) < c$$

となる c を求める。

H_0 が正しいときに, $\sqrt{n}(\overline{X}-\mu_0)/\sigma \sim N(0,1)$ となるので,

$$P\Bigl(\Bigl|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\Bigr| > z_{\alpha/2}\Bigr) = \alpha$$

すなわち,

$$P\Bigl(\exp\Bigl(-\frac{1}{2\sigma^2/n}(\overline{X}-\mu_0)^2\Bigr) < \exp\Bigl(-\frac{1}{2}z_{\alpha/2}^2\Bigr)\Bigr) = \alpha$$

と変形できる。したがって,

$$c = \exp\left(-\frac{1}{2}z_{\alpha/2}^2\right)$$

とすればよい。

例 2：　$X_1, X_n, \cdots, X_n$ は互いに独立で，それぞれパラメータ p を持ったベルヌイ分布に従うものとする。すなわち，X_i の確率関数は,

$$f(x;p) = p^x(1-p)^{1-x} \qquad x = 0, 1$$

となる。

このとき尤度関数は,

$$l(p) = \prod_{i=1}^{n} f(x_i;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

となる。

p の最尤推定量 $\widehat{p}$ は,

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

である。

次の仮説検定を考える。

$$H_0 :\ p = p_0 \qquad\qquad H_1 :\ p \neq p_0$$

→ 制約数は 1 つ。$(G = 1)$

尤度比は,

$$\frac{l(p_0)}{l(\widehat{p})} = \frac{\prod_{i=1}^{n} p_0^{X_i}(1-p_0)^{1-X_i}}{\prod_{i=1}^{n} \widehat{p}^{X_i}(1-\widehat{p})^{1-X_i}}$$

したがって，$n \longrightarrow \infty$ のとき,

$$-2\log\frac{l(p_0)}{l(\widehat{p})} = -2\log\frac{p_0}{\widehat{p}}\sum_{i=1}^{n} X_i - 2\log\frac{1-p_0}{1-\widehat{p}}\sum_{i=1}^{n}(1-X_i)$$

$\longrightarrow \chi^2(1)$

$\chi^2(1)$ 分布の上側 100 α% 点を $\chi^2_\alpha(1)$ とするとき，

$$-2\log\frac{l(p_0)}{l(\widehat{p})} > \chi^2_\alpha(1)$$

のとき，帰無仮説 H_0： $p = p_0$ を棄却する。

例 3： 回帰モデル

$$Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i$$

$$u_i \sim N(0, \sigma^2) \qquad i = 1, 2, \cdots, n$$

について，$\beta_1, \cdots, \beta_k$ に関する仮説の尤度比検定を行う。

例えば，

$$H_0： \beta_1 = 0$$

$$H_0:\ \beta_1+\beta_2=1$$

$$H_0:\ \beta_1=\beta_2=\beta_3=0$$

などのような仮説検定

$\theta=(\beta_1,\cdots,\beta_k,\sigma^2)$ とする。

尤度関数は,

$$\begin{aligned} l(\theta) &= \prod_{i=1}^{n} f(Y_i;\theta) \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\Bigl(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\beta_1X_{1i}-\cdots-\beta_kX_{ki})^2\Bigr) \end{aligned}$$

となる。

H_0 の制約つき最尤推定量を $\widetilde{\theta}=(\widetilde{\beta}_1,\cdots,\widetilde{\beta}_k,\widetilde{\sigma}^2)$ とする。この仮設に含まれる制約数を G とする。

制約なし最尤推定量を$\widehat{\theta} = (\widehat{\beta}_1, \cdots, \widehat{\beta}_k, \widehat{\sigma}^2)$ とする。

尤度比

$$\frac{l(\widetilde{\theta})}{l(\widehat{\theta})} = \frac{(2\pi\widetilde{\sigma}^2)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2\widetilde{\sigma}^2}\sum_{i=1}^{n}(Y_i - \widetilde{\beta}_1 X_{1i} - \cdots - \widetilde{\beta}_k X_{ki})^2\Big)}{(2\pi\widehat{\sigma}^2)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2\widehat{\sigma}^2}\sum_{i=1}^{n}(Y_i - \widehat{\beta}_1 X_{1i} - \cdots - \widehat{\beta}_k X_{ki})^2\Big)}$$

$$= \frac{(\widetilde{\sigma}^2)^{-\frac{n}{2}} \exp\Big(-\frac{n-G}{2}\Big)}{(\widehat{\sigma}^2)^{-\frac{n}{2}} \exp\Big(-\frac{n-k}{2}\Big)}$$

$$= \left(\frac{\dfrac{1}{n-G}\displaystyle\sum_{i=1}^{n}\widetilde{u}_i^2}{\dfrac{1}{n-k}\displaystyle\sum_{i=1}^{n}\widehat{u}_i^2}\right)^{-n/2} \exp\Big(-\frac{k-G}{2}\Big)$$

$$= \exp\Big(-\frac{k-G}{2}\Big)\left(\frac{n-k}{n-G}\right)^{-n/2}\left(\frac{\sum_{i=1}^{n}\widetilde{u}_i^2}{\sum_{i=1}^{n}\widehat{u}_i^2}\right)^{-n/2}$$

$$
\begin{aligned}
&= \exp\left(-\frac{k-G}{2}\right)\left(\frac{n-k}{n-G}\right)^{-n/2} \\
&\quad \times \left(1 + \frac{\sum_{i=1}^{n} \widetilde{u}_i^2 - \sum_{i=1}^{n} \widehat{u}_i^2}{\sum_{i=1}^{n} \widehat{u}_i^2}\right)^{-n/2} \\
&= \exp\left(-\frac{k-G}{2}\right)\left(\frac{n-k}{n-G}\right)^{-n/2} \\
&\quad \times \left(1 + \frac{G}{n-k}\frac{(\sum_{i=1}^{n} \widetilde{u}_i^2 - \sum_{i=1}^{n} \widehat{u}_i^2)/G}{\sum_{i=1}^{n} \widehat{u}_i^2/(n-k)}\right)^{-n/2} \\
&< c
\end{aligned}
$$

のとき仮説を棄却する。

$$
\frac{(\sum_{i=1}^{n} \widetilde{u}_i^2 - \sum_{i=1}^{n} \widehat{u}_i^2)/G}{\sum_{i=1}^{n} \widehat{u}_i^2/(n-k)} \sim F(G, n-k)
$$

を利用すると c が求まる。

ただし，途中で以下を利用

$$\widetilde{\sigma}^2 = \frac{1}{n-G}\sum_{i=1}^{n}(Y_i - \widetilde{\beta}_1 X_{1i} - \cdots - \widetilde{\beta}_k X_{ki})^2$$
$$= \frac{1}{n-G}\sum_{i=1}^{n}\widetilde{u}_i^2$$

$$\widehat{\sigma}^2 = \frac{1}{n-k}\sum_{i=1}^{n}(Y_i - \widehat{\beta}_1 X_{1i} - \cdots - \widehat{\beta}_k X_{ki})^2$$
$$= \frac{1}{n-k}\sum_{i=1}^{n}\widehat{u}_i^2$$

近似的には，

$$-2\log\frac{l(\widetilde{\theta})}{l(\widehat{\theta})} = -2\log\frac{(\widetilde{\sigma}^2)^{-\frac{n}{2}}\exp\left(-\dfrac{n-G}{2}\right)}{(\widehat{\sigma}^2)^{-\frac{n}{2}}\exp\left(-\dfrac{n-k}{2}\right)}$$

$$
\begin{aligned}
&= n\log\Big(\frac{\widetilde{\sigma}^2}{\widehat{\sigma}^2}\Big) + (k-G) \\
&\longrightarrow \chi^2(G)
\end{aligned}
$$

例 4： 回帰モデル

$$
\begin{aligned}
&Y_i = \alpha + \beta X_i + u_i \\
&u_i = \rho u_{i-1} + \epsilon_i \\
&\epsilon_i \sim N(0,\sigma^2) \qquad i = 2,3,\cdots,n
\end{aligned}
$$

について，H_0：$\rho = 0$，H_1：$\rho \neq 0$ の尤度比検定を行う。

$\theta = (\alpha,\beta,\sigma^2,\rho)$ とする。対数尤度関数は,

$$
\log l(\theta) = \sum_{i=2}^{n} \log f(Y_i;\theta) = -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\sigma^2)
$$

$$-\frac{1}{2\sigma^2}\sum_{i=2}^{n}\Big((Y_i-\rho Y_{i-1})-\alpha(1-\rho)-\beta(X_i-\rho X_{i-1})\Big)^2$$

となる。

対数尤度関数をそれぞれ α，β，σ^2，ρ について微分し，ゼロとおく。4 本の連立方程式を解いて，制約なし最尤推定量 $\widehat{\theta}=(\widehat{\alpha},\ \widehat{\beta},\ \widehat{\sigma}^2,\ \widehat{\rho})$ が得られる。

$\rho=0$ と制約をおく。$\theta=(\alpha,\beta,\sigma^2,0)$ とする。対数尤度関数は,

$$\log l(\theta)=\sum_{i=2}^{n}\log f(Y_i;\theta)=-\frac{n-1}{2}\log(2\pi)-\frac{n-1}{2}\log(\sigma^2)$$

$$-\frac{1}{2\sigma^2}\sum_{i=2}^{n}(Y_i-\alpha-\beta X_i)^2$$

となる。

上記の対数尤度関数をそれぞれ α，β，σ^2 について微分し，ゼロとおく。3 本の連立方程式を解いて，$\rho=0$ の制約付き最尤推定量 $\widetilde{\theta}=(\widetilde{\alpha},\ \widetilde{\beta},\ \widetilde{\sigma}^2,\ 0)$ が得られる。

すなわち，

$$\frac{\max_{\alpha,\beta,\sigma^2} l(\alpha,\beta,\sigma^2,0)}{\max_{\alpha,\beta,\sigma^2,\rho} l(\alpha,\beta,\sigma^2,\rho)} = \frac{l(\widetilde{\alpha},\widetilde{\beta},\widetilde{\sigma}^2,0)}{l(\widehat{\alpha},\widehat{\beta},\widehat{\sigma}^2,\widehat{\rho})} = \frac{l(\widetilde{\theta})}{l(\widehat{\theta})}$$

$\log l(\widehat{\theta})$ は，$\widehat{\sigma}^2 = \frac{1}{n-1}\sum_{i=2}^{n}\Big((Y_i - \widehat{\rho}Y_{i-1}) - \widehat{\alpha}(1-\widehat{\rho}) - \widehat{\beta}(X_i - \widehat{\rho}X_{i-1})\Big)^2$ に注意して，

$$\begin{aligned}\log l(\widehat{\theta}) &= -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\widehat{\sigma}^2) \\ &\quad -\frac{1}{2\widehat{\sigma}^2}\sum_{i=2}^{n}\Big((Y_i - \widehat{\rho}Y_{i-1}) - \widehat{\alpha}(1-\widehat{\rho}) - \widehat{\beta}(X_i - \widehat{\rho}X_{i-1})\Big)^2 \\ &= -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\widehat{\sigma}^2) - \frac{n-1}{2}\end{aligned}$$

となる。

同様に，$\log l(\widetilde{\theta})$ は，$\widetilde{\sigma}^2 = \frac{1}{n-1}\sum_{i=2}^{n}(Y_i - \widetilde{\alpha} - \widetilde{\beta}X_i)^2$ に注意して，

$$\log l(\widetilde{\theta}) = -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\widetilde{\sigma}^2)$$

$$
\begin{aligned}
&-\frac{1}{2\widetilde{\sigma}^2}\sum_{i=2}^{n}(Y_i-\widetilde{\alpha}-\widetilde{\beta}X_i)^2 \\
&=-\frac{n-1}{2}\log(2\pi)-\frac{n-1}{2}\log(\widetilde{\sigma}^2)-\frac{n-1}{2}
\end{aligned}
$$

となる。

したがって，尤度比検定統計量

$$
-2\log\frac{l(\widetilde{\theta})}{l(\widehat{\theta})}=(n-1)\log\frac{\widetilde{\sigma}^2}{\widehat{\sigma}^2}
$$

は，n が大きくなると，$\chi^2(1)$ 分布に近づく。

7 Time Series Analysis (時系列分析)

7.1 Introduction

1. **Stationarity (定常性) :**

   Let $y_1, y_2, \cdots, y_T$ be time series data.

   (a) **Weak Stationarity (弱定常性) :**

   $$\mathrm{E}(y_t) = \mu,$$

   $$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

   The first moment does not depend on time.

   The second moment depends only on time difference.

(b) **Strong Stationarity (強定常性) :**

Let $f(y_{t_1}, y_{t_2}, \cdots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \cdots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \cdots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \cdots, y_{t_r+\tau})$$

All the moments are same for all τ.

2. **Ergodicity (エルゴード性) :**

As time difference between two data is large, the two data become independent.

$y_1, y_2, \cdots, y_T$ is said to be ergodic in mean when $\overline{y}$ converges in probability to $\mathrm{E}(y_t)$.

3. **Auto-covariance Function (自己共分散関数) :**

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

$$\gamma(\tau) = \gamma(-\tau)$$

4. **Auto-correlation Function (自己相関関数) :**

$$\rho(\tau) = \frac{\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\mathrm{Var}(y_t)}\sqrt{\mathrm{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-\tau}) = \gamma(0)$.

5. **Sample Mean (標本平均) :**

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

6. **Sample Auto-covariance (標本自己共分散) :**

$$\hat{\gamma}(\tau) = \frac{1}{T}\sum_{t=\tau+1}^{T} (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

7. **Correlogram (コレログラム, or 標本自己相関関数) :**

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

8. **Lag Operator (ラグ作要素) :**

$$L^\tau y_t = y_{t-\tau}, \qquad \tau = 1, 2, \cdots$$

9. **Likelihood Function (尤度関数)** — Innovation Form :

The joint distribution of $y_1, y_2, \cdots, y_T$ is written as:

$$\begin{aligned} f(y_1, , y_2, \cdots, y_T) &= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}, \cdots, y_1) \\ &= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}|y_{T-2}, \cdots, y_1) f(y_{T-2}, \cdots, y_1) \\ &\ \vdots \\ &= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}|y_{T-2}, \cdots, y_1) \ \cdots \ f(y_2|y_1) f(y_1) \\ &= f(y_1) \prod_{t=2}^{T} f(y_t|y_{t-1}, \cdots, y_1). \end{aligned}$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \cdots, y_T) = \log f(y_1) + \sum_{t=2}^{T} \log f(y_t | y_{t-1}, \cdots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \cdots, y_1)$ is given by the normal distribution with conditional mean $\mathrm{E}(y_t|y_{t-1}, \cdots, y_1)$ and conditional variance $\mathrm{Var}(y_t|y_{t-1}, \cdot$

7.2 Time Series Models (時系列モデル)

Autoregressive Model (自己回帰モデル or AR モデル): AR(p)

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$

Moving Average Model (移動平均モデル or MA モデル): MA(q)

$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$

ARMA Model: ARMA(p, q)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

ARIMA Model: ARIMA(p, d, q)

$$\begin{aligned}
\Delta y_t &= y_t - y_{t-1} = (1 - L) y_t, \\
\Delta^2 y_t &= \Delta y_t - \Delta y_{t-1} = (1 - L)^2 y_t, \\
&\vdots \\
\Delta^d y_t &= (1 - L)^d y_t.
\end{aligned}$$

$$\Delta^d y_t \sim \text{ARMA}(p, q) \iff y_t \sim \text{ARIMA}(p, d, q)$$

$$\Delta^d y_t = \phi_1 \Delta^d y_{t-1} + \phi_2 \Delta^d y_{t-2} + \cdots + \phi_p \Delta^d y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

SARIMA Model: SARIMA(p, d, q)

${}^{s}\Delta y_t = y_t - y_{t-s}$, $s = 4$ for quarterly data $s = 12$ for monthly data

$${}^{s}\Delta\Delta^{d} y_t \ \sim \ \text{ARMA}(p, q) \iff y_t \ \sim \ \text{SARIMA}(p, d, q)$$

$${}^{s}\Delta\Delta^{d} y_t = \phi_1 \, {}^{s}\Delta\Delta^{d} y_{t-1} + \phi_2 \, {}^{s}\Delta\Delta^{d} y_{t-2} + \cdots + \phi_p \, {}^{s}\Delta\Delta^{d} y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

7.3 Autoregressive Model (自己回帰モデル or AR モデル)

1. **AR(p) Model :**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. **Stationarity (定常性) :**

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one in absolute value, y_t is stationary.

Suppose that the p solutions include imaginary numbers.

When the p solutions are outside unit circle, y_t is stationary.

3. Remark for **Partial Autocorrelation Coefficient (偏自己相関係数), $\boldsymbol{\phi_{k,k}}$:**

AR(p) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t.$$

Multiplying y_{t-i} on both sides, we have:

$$y_t y_{t-i} = \phi_1 y_{t-1} y_{t-i} + \phi_2 y_{t-2} y_{t-i} + \cdots + \phi_p y_{t-p} y_{t-i} + \epsilon_t y_{t-i},$$

for $i = 1, 2, \cdots, p$. Taking the expectation on both sides, we obtain:

$$\mathrm{E}(y_t y_{t-i}) = \phi_1 \mathrm{E}(y_{t-1} y_{t-i}) + \phi_2 \mathrm{E}(y_{t-2} y_{t-i}) + \cdots + \phi_p \mathrm{E}(y_{t-p} y_{t-i}) + \mathrm{E}(\epsilon_t y_{t-i}),$$

for $i = 1, 2, \cdots, p$.

Noting $\mathrm{E}(y_t y_{t-i}) = \gamma(i)$ and $\mathrm{E}(\epsilon_t y_{t-i}) = 0$ for $i = 1, 2, \cdots, p$, we obtain:

$$\gamma(i) = \phi_1 \gamma(i-1) + \phi_2 \gamma(i-2) + \cdots + \phi_p \gamma(i-p),$$

for $i = 1, 2, \cdots, p$.

Noting $\mathrm{E}(y_t y_s) = \gamma(t-s)$, we obtain:

$$\begin{aligned}
\gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(-1) + \cdots + \phi_p \gamma(1-p), \\
\gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0) + \cdots + \phi_p \gamma(2-p), \\
&\vdots \\
\gamma(p) &= \phi_1 \gamma(p-1) + \phi_2 \gamma(p-2) + \cdots + \phi_p \gamma(0).
\end{aligned}$$

From $\gamma(\tau) = \gamma(-\tau)$, we have:

$$
\begin{aligned}
\gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_p\gamma(p-1), \\
\gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \cdots + \phi_p\gamma(p-2), \\
&\vdots \\
\gamma(p) &= \phi_1\gamma(p-1) + \phi_2\gamma(p-2) + \cdots + \phi_p\gamma(0).
\end{aligned}
$$

Using the matrix form, we obtain:

$$
\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{pmatrix}
=
\begin{pmatrix}
\gamma(0) & \gamma(1) & \cdots & \gamma(p-1) \\
\gamma(1) & \gamma(0) & \cdots & \gamma(p-2) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(p-1) & \gamma(p-2) & \cdots & \gamma(0)
\end{pmatrix}
\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix}
$$

4. **Partial Autocorrelation Coefficient (偏自己相関係数), $\boldsymbol{\phi_{k,k}}$:**

The partial autocorrelation coefficient between y_t and y_{t-k}, denoted by $\phi_{k,k}$, is a measure of strength of the relationship between y_t and y_{t-k}, after removing influence of $y_{t-1}, \cdots, y_{t-k+1}$.

$$\phi_{1,1} = \rho(1)$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{pmatrix}$$

$$\vdots$$

$$
\begin{pmatrix}
1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\
\rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\
\vdots & \vdots & & \vdots & \vdots \\
\rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1
\end{pmatrix}
\begin{pmatrix}
\phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k}
\end{pmatrix}
=
\begin{pmatrix}
\rho(1) \\ \rho(2) \\ \vdots \\ \rho(k)
\end{pmatrix}
$$

Use Cramer's rule (クラメールの公式) to obtain $\phi_{k,k}$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

2. Rewriting the AR(1) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\
&\vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.
\end{aligned}
$$

As s is large, ϕ_1^s approaches zero. $\Longrightarrow$ Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$
y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots
$$

MA representation of AR model.

(MA will be discussed later.)

4. Mean of AR(1) process, μ

$$\begin{aligned}\mu = \mathrm{E}(y_t) &= \mathrm{E}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots) \\ &= \mathrm{E}(\epsilon_t) + \phi_1 \mathrm{E}(\epsilon_{t-1}) + \phi_1^2 \mathrm{E}(\epsilon_{t-2}) + \cdots = 0\end{aligned}$$

5. Autocovariance and autocorrelation functions of the AR(1) process:

Rewriting the AR(1) process, we have:

$$y_t = \phi_1^{\tau} y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\ &= \mathrm{E}\Big((\phi_1^{\tau} y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}) y_{t-\tau}\Big) \\ &= \phi_1^{\tau} \mathrm{E}(y_{t-\tau} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) + \phi_1 \mathrm{E}(\epsilon_{t-1} y_{t-\tau}) + \cdots + \phi_1^{\tau-1} \mathrm{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\ &= \phi_1^{\tau} \gamma(0).\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^\tau.$$

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$\mathrm{E}(y_t y_{t-\tau}) = \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2 = \phi_1^2 \gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1 \gamma(0)$.

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2}$$

6. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2)-\rho(1)^2}{1-\rho(1)^2} = 0$$

7. Estimation of AR(1) model:

   (a) Likelihood function

$$\log f(y_T,\cdots,y_1) = \log f(y_1) + \sum_{t=1}^{T} \log f(y_t|y_{t-1},\cdots,y_1)$$

$$= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi_1^2}\right) - \frac{1}{\sigma^2/(1-\phi_1^2)}y_1^2$$

$$-\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right)$$

$$-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

Note as follows:

$$f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)$$

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)$$