

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

Test  $H_0 : \phi_1 = 1$  against  $H_1 : \phi_1 < 1$ .

Equivalently,

$$\Delta y_t = \rho y_{t-1} + \epsilon_t.$$

Test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ .

### ***t* Distribution**

<i>T</i>	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-2.49	-2.06	-1.71	-1.32	1.32	1.71	2.06	2.49
50	-2.40	-2.01	-1.68	-1.30	1.30	1.68	2.01	2.40
100	-2.36	-1.98	-1.66	-1.29	1.29	1.66	1.98	2.36
250	-2.34	-1.97	-1.65	-1.28	1.28	1.65	1.97	2.34
500	-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33
$\infty$	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33

$$(a) H_0 : y_t = y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

$T$	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-2.66	-2.26	-1.95	-1.60	0.92	1.33	1.70	2.16
50	-2.62	-2.25	-1.95	-1.61	0.91	1.31	1.66	2.08
100	-2.60	-2.24	-1.95	-1.61	0.90	1.29	1.64	2.03
250	-2.58	-2.23	-1.95	-1.62	0.89	1.29	1.63	2.01
500	-2.58	-2.23	-1.95	-1.62	0.89	1.28	1.62	2.00
$\infty$	-2.58	-2.23	-1.95	-1.62	0.89	1.28	1.62	2.00

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.

$$(b) H_0 : y_t = y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

$T$	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-3.75	-3.33	-3.00	-2.63	-0.37	0.00	0.34	0.72
50	-3.58	-3.22	-2.93	-2.60	-0.40	-0.03	0.29	0.66
100	-3.51	-3.17	-2.89	-2.58	-0.42	-0.05	0.26	0.63
250	-3.46	-3.14	-2.88	-2.57	-0.42	-0.06	0.24	0.62
500	-3.44	-3.13	-2.87	-2.57	-0.43	-0.07	0.24	0.61
$\infty$	-3.43	-3.12	-2.86	-2.57	-0.44	-0.07	0.23	0.60

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \alpha_0 + \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.

$$(c) H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

$T$	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-4.38	-3.95	-3.60	-3.24	-1.14	-0.80	-0.50	-0.15
50	-4.15	-3.80	-3.50	-3.18	-1.19	-0.87	-0.58	-0.24
100	-4.04	-3.73	-3.45	-3.15	-1.22	-0.90	-0.62	-0.28
250	-3.99	-3.69	-3.43	-3.13	-1.23	-0.92	-0.64	-0.31
500	-3.98	-3.68	-3.42	-3.13	-1.24	-0.93	-0.65	-0.32
$\infty$	-3.96	-3.66	-3.41	-3.12	-1.25	-0.94	-0.66	-0.33

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \alpha_0 + \alpha_1 t + \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.

## 9.2 Serially Correlated Errors

Consider the case where the error term is serially correlated.

### 9.2.1 Augmented Dickey-Fuller (ADF) Test

Consider the following AR( $p$ ) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \quad \epsilon_t \sim \text{iid}(0, \sigma_\epsilon^2),$$

which is rewritten as:  $\phi(L)y_t = \epsilon_t$ .

When the above model has a unit root, we have  $\phi(1) = 0$ , i.e.,  $\phi_1 + \phi_2 + \cdots + \phi_p = 1$ .

The above AR( $p$ ) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + \delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where  $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$  and  $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$ .

The null and alternative hypotheses are:

$$H_0 : \rho = 1 \text{ (Unit root),}$$

$$H_1 : \rho < 1 \text{ (Stationary).}$$

Use the  $t$  test, where we have the same asymptotic distributions.

We can utilize the same tables as before.

Choose  $p$  by AIC or SBIC.

Use  $N(0, 1)$  to test  $H_0 : \delta_j = 0$  against  $H_1 : \delta_j \neq 0$  for  $j = 1, 2, \dots, p - 1$ .

## Reference

Kurozumi (2008) “Economic Time Series Analysis and Unit Root Tests: Development and Perspective,” *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.

Download the above paper from:

[http://ci.nii.ac.jp/vol\\_issue/nels/AA11989749/ISS0000426576\\_ja.html](http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html)

## Example of ADF Test

```
. gen time=_n
. tsset time
    time variable:  time, 1 to 516
                delta: 1 unit
. gen sexpend=expnd-112.expnd
(12 missing values generated)
. corrgram sexpend
```

LAG	AC	PAC	Q	Prob>Q	<sup>-1</sup> [Autocorrelation]	<sup>0</sup> [Partial Autocor]
1	0.7177	0.7184	261.14	0.0000	-----	-----
2	0.7036	0.3895	512.6	0.0000	-----	---
3	0.7031	0.2817	764.23	0.0000	-----	--
4	0.6366	0.0456	970.94	0.0000	-----	
5	0.6413	0.1116	1181.1	0.0000	-----	
6	0.6267	0.0815	1382.2	0.0000	-----	
7	0.6208	0.0972	1580	0.0000	-----	
8	0.6384	0.1286	1789.5	0.0000	-----	-
9	0.5926	-0.0205	1970.5	0.0000	-----	
10	0.5847	-0.0014	2146.9	0.0000	-----	
11	0.5658	-0.0185	2312.6	0.0000	-----	
12	0.4529	-0.2570	2418.9	0.0000	-----	--
13	0.5601	0.2318	2581.8	0.0000	-----	-
14	0.5393	0.1095	2733.2	0.0000	-----	
15	0.5277	0.0850	2878.4	0.0000	-----	

. varsoc d.sexpend, exo(1.sexpend) maxlag(25)

Selection-order criteria

Sample: 39 - 516

Number of obs

=

478

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	-4917.7				5.1e+07	20.5845	20.5914	20.6019
1	-4878.69	78.013	1	0.0000	4.3e+07	20.4255	20.4358	20.4516
2	-4858.95	39.481	1	0.0000	4.0e+07	20.3471	20.3608	20.382
3	-4858.46	.97673	1	0.323	4.0e+07	20.3492	20.3664	20.3928
4	-4855.44	6.0461	1	0.014	4.0e+07	20.3407	20.3613	20.3931
5	-4853.84	3.1904	1	0.074	4.0e+07	20.3383	20.3623	20.3993
6	-4851.58	4.5304	1	0.033	4.0e+07	20.333	20.3604	20.4027
7	-4847.61	7.942	1	0.005	3.9e+07	20.3205	20.3514	20.399
8	-4847.51	.20154	1	0.653	3.9e+07	20.3243	20.3586	20.4115
9	-4847.51	.00096	1	0.975	3.9e+07	20.3285	20.3662	20.4244
10	-4847.43	.16024	1	0.689	4.0e+07	20.3323	20.3735	20.437
11	-4831.38	32.094	1	0.000	3.7e+07	20.2694	20.3139	20.3828
12	-4818.46	25.834	1	0.000	3.5e+07	20.2195	20.2675	20.3416*
13	-4815.64	5.6341	1	0.018	3.5e+07	20.2119	20.2633	20.3427
14	-4813.98	3.321	1	0.068	3.5e+07	20.2091	20.264	20.3487
15	-4813.38	1.2007	1	0.273	3.5e+07	20.2108	20.2691	20.3591
16	-4810.57	5.6184	1	0.018	3.5e+07	20.2032	20.265	20.3603
17	-4808.7	3.7539	1	0.053	3.5e+07	20.1996	20.2647	20.3653
18	-4806.12	5.1557	1	0.023	3.4e+07	20.193	20.2616	20.3674
19	-4804.6	3.0319	1	0.082	3.4e+07	20.1908	20.2628	20.374
20	-4804.6	2.7e-05	1	0.996	3.5e+07	20.195	20.2704	20.3869
21	-4797.33	14.542	1	0.000	3.4e+07	20.1688	20.2476	20.3694
22	-4794.2	6.2571*	1	0.012	3.3e+07*	20.1598*	20.2422*	20.3692
23	-4793.42	1.5626	1	0.211	3.3e+07	20.1608	20.2465	20.3788
24	-4792.85	1.1533	1	0.283	3.3e+07	20.1625	20.2517	20.3893

```
| 25 | -4792.78 .13518 1 0.713 3.4e+07 20.1664 20.259 20.402 |
+-----+
Endogenous: D.sexpend
Exogenous: L.sexpend _cons
```

```
. dfuller sexpend, lags(23)
```

```
Augmented Dickey-Fuller test for unit root          Number of obs   =      480
```

```

----- Interpolated Dickey-Fuller -----
          Test          1% Critical    5% Critical    10% Critical
          Statistic      Value          Value          Value
-----
Z(t)          -1.754          -3.442          -2.871          -2.570
-----
```

```
MacKinnon approximate p-value for Z(t) = 0.4033
```

```
. dfuller sexpend, lags(13)
```

```
Augmented Dickey-Fuller test for unit root          Number of obs   =      490
```

```

----- Interpolated Dickey-Fuller -----
          Test          1% Critical    5% Critical    10% Critical
          Statistic      Value          Value          Value
-----
Z(t)          -2.129          -3.441          -2.870          -2.570
-----
```

```
MacKinnon approximate p-value for Z(t) = 0.2329
```

```
.
```

## 9.3 Cointegration (共和分)

1. For a scalar  $y_t$ , when  $\Delta y_t = y_t - y_{t-1}$  is a white noise (i.e., iid), we write  $\Delta y_t \sim I(1)$ .

### 2. Definition of Cointegration:

Suppose that each series in a  $g \times 1$  vector  $y_t$  is  $I(1)$ , i.e., each series has unit root, and that a linear combination of each series (i.e.  $a'y_t$  for a nonzero vector  $a$ ) is  $I(0)$ , i.e., stationary.

Then, we say that  $y_t$  has a cointegration.

### 3. Example:

Suppose that  $y_t = (y_{1,t}, y_{2,t})'$  is the following vector autoregressive process:

$$y_{1,t} = \phi_1 y_{2,t} + \epsilon_{1,t},$$

$$y_{2,t} = y_{2,t-1} + \epsilon_{2,t}.$$

Then,

$$\Delta y_{1,t} = \phi_1 \epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \quad (\text{MA}(1) \text{ process}),$$

$$\Delta y_{2,t} = \epsilon_{2,t},$$

where both  $y_{1,t}$  and  $y_{2,t}$  are  $I(1)$  processes.

The linear combination  $y_{1,t} - \phi_1 y_{2,t}$  is  $I(0)$ .

In this case, we say that  $y_t = (y_{1,t}, y_{2,t})'$  is cointegrated with  $a = (1, -\phi_1)$ .

$a = (1, -\phi_1)$  is called the cointegrating vector, which is not unique.

Therefore, the first element of  $a$  is set to be one.

4. Suppose that  $y_t \sim I(1)$  and  $x_t \sim I(1)$ .

For the regression model  $y_t = x_t \beta + u_t$ , OLS does not work well if we do not have the  $\beta$  which satisfies  $u_t \sim I(0)$ .

⇒ **Spurious regression** (見せかけの回帰)

5. Suppose that  $y_t \sim I(1)$ ,  $y_t$  is a  $g \times 1$  vector and  $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$ .

$y_{2,t}$  is a  $k \times 1$  vector, where  $k = g - 1$ .

Consider the following regression model:

$$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t, \quad t = 1, 2, \dots, T.$$

OLSE is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t}y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t}y_{2,t} \end{pmatrix}.$$

Next, consider testing the null hypothesis  $H_0 : R\gamma = r$ , where  $R$  is a  $m \times k$  matrix ( $m \leq k$ ) and  $r$  is a  $m \times 1$  vector.

The  $F$  statistic, denoted by  $F_T$ , is given by:

$$F_T = \frac{1}{m} (R\hat{\gamma} - r)' \left( s_T^2 \begin{pmatrix} 0 & R \end{pmatrix} \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t}y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R' \end{pmatrix} \right)^{-1} (R\hat{\gamma} - r),$$

where

$$s_T^2 = \frac{1}{T - g} \sum_{t=1}^T (y_{1,t} - \hat{\alpha} - \hat{\gamma}'y_{2,t})^2.$$

When we have the  $\gamma$  such that  $y_{1,t} - \gamma y_{2,t}$  is stationary, OLSE of  $\gamma$ , i.e.,  $\hat{\gamma}$ , is not statistically equal to zero.

When the sample size  $T$  is large enough,  $H_0$  is rejected by the  $F$  test.

6. Phillips, P.C.B. (1986) "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, Vol.33, pp.95 – 131.

Consider a  $g \times 1$  vector  $y_t$  whose first difference is described by:

$$\Delta y_t = \Psi(L)\epsilon_t = \sum_{s=0}^{\infty} \Psi_s \epsilon_{t-s},$$

for  $\epsilon_t$  an i.i.d.  $g \times 1$  vector with mean zero, variance  $E(\epsilon_t \epsilon_t') = PP'$ , and finite fourth moments and where  $\{s\Psi_s\}_{s=0}^{\infty}$  is absolutely summable.

Let  $k = g - 1$  and  $\Lambda = \Psi(1)P$ .

Partition  $y_t$  as  $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$  and  $\Lambda\Lambda'$  as  $\Lambda\Lambda' = \begin{pmatrix} \Sigma_{11} & \Sigma'_{21} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ , where  $y_{1,t}$  and  $\Sigma_{11}$  are scalars,  $y_{2,t}$  and  $\Sigma_{21}$  are  $k \times 1$  vectors, and  $\Sigma_{22}$  is a  $k \times k$  matrix.

Suppose that  $\Lambda\Lambda'$  is nonsingular, and define  $\sigma_1^{*2} = \Sigma_{11} - \Sigma'_{21}\Sigma_{22}^{-1}\Sigma_{21}$ .

Let  $L_{22}$  denote the Cholesky factor of  $\Sigma_{22}^{-1}$ , i.e.,  $L_{22}$  is the lower triangular matrix satisfying  $\Sigma_{22}^{-1} = L_{22}L'_{22}$ .

Then, (a) – (c) hold.

(a) OLSEs of  $\alpha$  and  $\gamma$  in the regression model  $y_{1,t} = \alpha + \gamma'y_{2,t} + u_t$ , denoted by  $\hat{\alpha}_T$  and  $\hat{\gamma}_T$ , are characterized by:

$$\begin{pmatrix} T^{-1/2}\hat{\alpha}_T \\ \hat{\gamma}_T - \Sigma_{22}^{-1}\Sigma_{21} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1^* h_1 \\ \sigma_1^* L_{22} h_2 \end{pmatrix},$$

where 
$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & \int_0^1 W_2^*(r)' dr \\ \int_0^1 W_2^*(r) dr & \int_0^1 W_2^*(r) W_2^*(r)' dr \end{pmatrix}^{-1} \begin{pmatrix} \int_0^1 W_1^*(r) dr \\ \int_0^1 W_2^*(r) W_1^*(r) dr \end{pmatrix}.$$

$W_1^*(r)$  and  $W_2^*(r)$  denote scalar and  $g$ -dimensional standard Brownian motions, and  $W_1^*(r)$  is independent of  $W_2^*(r)$ .

(b) The sum of squared residuals, denoted by  $RSS_T = \sum_{t=1}^T \hat{u}_t^2$ , satisfies

$$T^{-2}RSS_T \rightarrow \sigma_1^{*2}H,$$

where 
$$H = \int_0^1 (W_1^*(r))^2 dr - \left( \begin{pmatrix} \int_0^1 W_1^*(r) dr \\ \int_0^1 W_2^*(r) W_1^*(r) dr \end{pmatrix}' \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right)^{-1}.$$

(c) The  $F_T$  test satisfies:

$$\begin{aligned} T^{-1}F_T &\rightarrow \frac{1}{m}(\sigma_1^*R^*h_2 - r^*)' \\ &\times \left( \sigma_1^{*2}H \begin{pmatrix} 0 & R^* \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W_2^*(r)' dr \\ \int_0^1 W_2^*(r) dr & \int_0^1 W_2^*(r) W_2^*(r)' dr \end{pmatrix}^{-1} \begin{pmatrix} 0 & R^* \end{pmatrix}' \right)^{-1} \\ &\times (\sigma_1^*R^*h_2 - r^*), \end{aligned}$$

where  $R^* = RL_{22}$  and  $r^* = r - R\Sigma_{22}^{-1}\Sigma_{21}$ .

## Summary:

(a) indicates that OLSE  $\hat{\gamma}_T$  is not consistent.

(b) indicates that  $s_T^2 = \frac{1}{T-g} \sum_{t=1}^T \hat{u}_t^2$  diverges.

(c) indicates that  $F_T$  diverges.

⇒ **Spurious regression** (見せかけの回帰)