Testing Linear Restrictions (F Distribution):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R: \ G \times k, \qquad \text{rank}(R) = G \leq k.$

   $R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$.

   Therefore, $\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(G)$.

   Note that $R\beta = r$.

   (a) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the mean of $R\hat{\beta}$ is:

   $$\mathrm{E}(R\hat{\beta}) = R\mathrm{E}(\hat{\beta}) = R\beta.$$

   (b) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the variance of $R\hat{\beta}$ is:

$$
\begin{aligned}
\mathrm{V}(R\hat{\beta}) &= \mathrm{E}((R\hat{\beta}-R\beta)(R\hat{\beta}-R\beta)') = \mathrm{E}(R(\hat{\beta}-\beta)(\hat{\beta}-\beta)'R') \\
&= R\mathrm{E}((\hat{\beta}-\beta)(\hat{\beta}-\beta)')R' = R\mathrm{V}(\hat{\beta})R' = \sigma^2 R(X'X)^{-1}R'.
\end{aligned}
$$

2. We know that $\dfrac{(n-k)s^2}{\sigma^2} = \dfrac{e'e}{\sigma^2} = \dfrac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\sigma^2} \sim \chi^2(n-k)$.

3. Under normality assumption on u, $\hat{\beta}$ is independent of e.

4. Therefore, we have the following distribution:

$$
\frac{(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)/G}{(y-X\hat{\beta})'(y-X\hat{\beta})/(n-k)} \sim F(G, n-k)
$$

5. **Some Examples:**

(a) t Test:

The case of $G = 1$, $r = 0$ and $R = (0, \cdots, 1, \cdots, 0)$ (the ith element of R is one and the other elements are zero):

The test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{s^2} = \frac{\hat{\beta}_i^2}{s^2 a_{ii}} \sim F(1, n-k),$$

where $s^2 = e'e/(n-k)$, $R\hat{\beta} = \hat{\beta}_i$ and

$$a_{ii} = R(X'X)^{-1}R' = \text{ the } i \text{ row and } i\text{th column of } (X'X)^{-1}.$$

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $H_0: \ \beta_i = 0$ is given by:

$$\frac{\hat{\beta}_i}{s\sqrt{a_{ii}}} \sim t(n-k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m+1, m+2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the F test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \text{rank}(R) = k$ and β is a $2k \times 1$ vector.

The distribution is $F(k, n - 2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \mathrm{rank}(R) = 1$

The distribution of the test statistic is $F(1, n-k)$.

(d) Testing seasonality:

In the case of **quarterly data (四半期データ)**, the regression model is:

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j = 1$ in the jth quarter and 0 otherwise, i.e., D_j, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\Longrightarrow H_0: \ \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \mathrm{rank}(R) = 3$, and β is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n-k)$.

(e) Cobb-Douglas Production Function:

Let Q_i, K_i and L_i be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We test a linear homogeneous (一次同次) production function.

The null and alternative hypotheses are:

$$H_0: \ \beta_2 + \beta_3 = 1,$$

$$H_1: \ \beta_2 + \beta_3 \neq 1.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods m and $m+1$.

In the case where both the constant term and the slope are changed, the regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m+1, m+2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m+1$.

The null and alternative hypotheses are as follows:

$H_0: \ \gamma = \delta = 0,$

$H_1: \ \gamma \neq 0, \text{ or, } \delta \neq 0.$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither x_i nor z_i depends on y_i.

In this case, the null and alternative hypotheses are as follows:

$$H_0 : \ \beta = \gamma = 0,$$

$$H_1 : \ \beta \neq 0, \ \text{or}, \ \gamma \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Coefficient of Determination R^2 and F distribution:

- The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = \begin{pmatrix} 1 & x_{2i} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$x_i: \ 1 \times k, \qquad x_{2i}: \ 1 \times (k-1), \qquad \beta: \ k \times 1, \qquad \beta_2: \ (k-1) \times 1$

Define:

$$X_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Then,

$$y = X\beta + u = \begin{pmatrix} i & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u = i\beta_1 + X_2\beta_2 + u,$$

where the first column of X corresponds to a constant term, i.e.,

$$X = (\, i \quad X_2 \,), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- Consider testing $H_0 : \ \beta_2 = 0$.

The F distribution is set as follows:

$$R = (\, 0 \quad I_{k-1} \,), \qquad r = 0$$

where R is a $(k-1) \times k$ matrix and r is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

We are going to show:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \frac{1}{n} i i'$.

Note that M is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$\begin{aligned} R(X'X)^{-1}R' &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} \begin{pmatrix} i & X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \end{aligned}$$

[Review] The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.

[End of Review]

Go back to the F distribution.

$$\begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix}$$

Therefore, we obtain:

$$\begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}.$$

Thus, under $H_0: \ \beta_2 = 0$, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k).$$

● Coefficient of Determination R^2:

Define e as $e = y - X\hat{\beta}$. The coefficient of determinant, R^2, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \frac{1}{n}ii'$, I_n is a $n \times n$ identity matrix and i is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = \begin{pmatrix} i & X_2 \end{pmatrix}$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

because $i'e = 0$, and

$$MX = M\begin{pmatrix} i & X_2 \end{pmatrix} = \begin{pmatrix} Mi & MX_2 \end{pmatrix} = \begin{pmatrix} 0 & MX_2 \end{pmatrix},$$

because $Mi = 0$.

$$MX\hat{\beta} = (\,0 \quad MX_2\,)\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = MX_2\hat{\beta}_2.$$

Thus,

$$My = MX\hat{\beta} + Me \qquad \Longrightarrow \qquad My = MX_2\hat{\beta}_2 + e.$$

$y'My$ is given by: $y'My = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 + e'e$, because $X_2'e = 0$ and $Me = e$.

The coefficient of determinant, R^2, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \qquad \Longrightarrow \qquad e'e = (1 - R^2)y'My,$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My} \qquad \Longrightarrow \qquad \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 = R^2y'My.$$

Therefore,

$$\frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} = \frac{R^2y'My/(k-1)}{(1-R^2)y'My/(n-k)} = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k).$$

Thus, using R^2, the null hypothesis $H_0:\ \beta_2 = 0$ is easily tested.