# 5 Restricted OLS (制約付き最小二乗法)

1. Let $\tilde{\beta}$ be the restricted estimator.

   Consider the linear restriction: $R\beta = r$.

2. Minimize $(y - X\tilde{\beta})'(y - X\tilde{\beta})$ subject to $R\tilde{\beta} = r$.

   Let $L$ be the Lagrangian for the minimization problem.

$$L = (y - X\tilde{\beta})'(y - X\tilde{\beta}) - 2\tilde{\lambda}'(R\tilde{\beta} - r)$$

   Let $\tilde{\beta}$ and $\tilde{\lambda}$ be the solutions of $\beta$ and $\lambda$ in the optimization problem shown above.

   That is, $\tilde{\beta}$ and $\tilde{\lambda}$ minimize the Lagrangian $L$.

Therefore, we solve the following equations:

$$\frac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0$$
$$\frac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0.$$

(*) Remember that $\dfrac{\partial a'x}{\partial x} = a$ and $\dfrac{\partial x'Ax}{\partial x} = (A + A')x$.

From $\dfrac{\partial L}{\partial \tilde{\beta}} = 0$, we obtain:

$$\tilde{\beta} = (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}.$$

Multiplying $R$ from the left, we have:

$$R\tilde{\beta} = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Because $R\tilde{\beta} = r$ has to be satisfied, we have the following expression:

$$r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Therefore, solving the above equation with respect to $\tilde{\lambda}$, we obtain:

$$\tilde{\lambda} = \left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta})$$

Substituting $\tilde{\lambda}$ into $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}$, the restricted OLSE is given by:

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}).$$

(a) The expectation of $\tilde{\beta}$ is:

$$\begin{aligned}
\mathrm{E}(\tilde{\beta}) &= \mathrm{E}(\hat{\beta}) + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\mathrm{E}(\hat{\beta})) \\
&= \beta + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\beta) \\
&= \beta,
\end{aligned}$$

because of $R\beta = r$.

Thus, it is shown that $\tilde{\beta}$ is unbiased.

(b) The variance of $\tilde{\beta}$ is as follows.

First, rewrite as follows:

$$
\begin{aligned}
(\tilde{\beta} - \beta) &= (\hat{\beta} - \beta) + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\beta - R\hat{\beta}) \\
&= (\hat{\beta} - \beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - R\beta) \\
&= (\hat{\beta} - \beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(\hat{\beta} - \beta) \\
&= \left(I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\right)(\hat{\beta} - \beta) \\
&= W(\hat{\beta} - \beta),
\end{aligned}
$$

where $W \equiv I_k - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R$.

Then, we obtain the following variance:

$$
\begin{aligned}
\mathrm{V}(\tilde{\beta}) &\equiv \mathrm{E}((\tilde{\beta}-\beta)(\tilde{\beta}-\beta)') = \mathrm{E}(W(\hat{\beta}-\beta)(\hat{\beta}-\beta)'W') \\
&= W\mathrm{E}((\hat{\beta}-\beta)(\hat{\beta}-\beta)')W' = W\mathrm{V}(\hat{\beta})W' = \sigma^2 W(X'X)^{-1}W' \\
&= \sigma^2\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)(X'X)^{-1} \\
&\qquad \times\Big(I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)' \\
&= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1} \\
&= \mathrm{V}(\hat{\beta}) - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}
\end{aligned}
$$

That is,

$$
\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta}) = \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}
$$

Thus, $\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta})$ is positive definite.

If $X'X$ is positive definite,

$\Longrightarrow$ then $(X'X)^{-1}$ is also positive definite,

$\Longrightarrow$ then $R(X'X)^{-1}R'$ is also positive definite,

$\Longrightarrow$ then $(R(X'X)^{-1}R')^{-1}$ is also positive definite,

$\Longrightarrow$ then $(X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}R(X'X)^{-1}$ is also positive definite,

Let $a$ be a $k \times 1$ vector.

Defining $z = Xa$, which is a $n \times 1$ vector, construct the sum of squared elements $z'z = \sum_{i=1}^{n} z_i^2 > 0$ for $z \neq 0$.

Threfore, we obtain: $z'z = (Xa)'(Xa) = a'X'Xa > 0$ for $z = Xa \neq 0$.

Thus, $X'X$ is positive definite.

3. Another solution:

Again, write the first-order condition for minimization:

$$\frac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0,$$
$$\frac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0,$$

which can be written as:

$$X'X\tilde{\beta} - R'\tilde{\lambda} = X'y,$$
$$R\tilde{\beta} = r.$$

Using the matrix form:

$$\begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix} \begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'y \\ r \end{pmatrix}.$$

The solutions of $\tilde{\beta}$ and $-\tilde{\lambda}$ are given by:

$$\begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1} \begin{pmatrix} X'y \\ r \end{pmatrix}.$$

(*) Formula to the inverse matrix:

$$\begin{pmatrix} A & B \\ B' & D \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ F' & G \end{pmatrix},$$

where $E$, $F$ and $G$ are given by:

$$E = (A - BD^{-1}B')^{-1} = A^{-1} + A^{-1}B(D - B'A^{-1}B)^{-1}B'A^{-1}$$

$$F = -(A - BD^{-1}B')^{-1}BD^{-1} = -A^{-1}B(D - B'A^{-1}B)^{-1}$$

$$G = (D - B'A^{-1}B)^{-1} = D^{-1} + D^{-1}B'(A - BD^{-1}B')^{-1}BD^{-1}$$

In this case, $E$ and $F$ correspond to:

$$E = (X'X)^{-1} - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}$$
$$F = (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}.$$

Therefore, $\tilde{\beta}$ is derived as follows:

$$\begin{aligned}\tilde{\beta} &= EX'y + Fr \\ &= \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}).\end{aligned}$$

The variance is:

$$\mathrm{V}\begin{pmatrix}\tilde{\beta} \\ -\tilde{\lambda}\end{pmatrix} = \sigma^2\begin{pmatrix}X'X & R' \\ R & 0\end{pmatrix}^{-1}.$$

Therefore, $\mathrm{V}(\tilde{\beta})$ is:

$$\mathrm{V}(\tilde{\beta}) = \sigma^2 E = \sigma^2\left((X'X)^{-1} - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}\right)$$

Under the restriction: $R\beta = r$,

$$\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta}) = \sigma^2 (X'X)^{-1} R' \left( R(X'X)^{-1} R' \right)^{-1} R(X'X)^{-1}$$

is positive definite.

# 6 $F$ Distribution (Restricted and Unrestricted OLSs)

1. As mentioned above, under the null hypothesis $H_0 : R\beta = r$,

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(G, n-k),$$

where $G = \text{Rank}(R)$.

Using $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta})$, the numerator is rewritten as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = (\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta}).$$

Moreover, the numerator is represented as follows:

$$\begin{aligned}(y - X\tilde{\beta})'(y - X\tilde{\beta}) &= (y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}))'(y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}))\\ &= (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta})\end{aligned}$$

$$-(y - X\hat{\beta})'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'(y - X\hat{\beta})$$

$$=(y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}).$$

$X'(y - X\hat{\beta}) = X'e = 0$ is utilized.

Summarizing, we have following representation:

$$\begin{aligned}(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) &= (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\ &= (y - X\tilde{\beta})'(y - X\tilde{\beta}) - (y - X\hat{\beta})'(y - X\hat{\beta}) \\ &= \tilde{u}'\tilde{u} - e'e,\end{aligned}$$

where $e$ and $\tilde{u}$ are the restricted residual and the unrestricted residual, i.e., $e = y - X\hat{\beta}$ and $\tilde{u} = y - X\tilde{\beta}$.

Therefore, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n - k)} = \frac{(\tilde{u}'\tilde{u} - e'e)/G}{e'e/(n - k)} \sim F(G, n - k).$$