Solutions of the homework #2

LU ANG

April 24th, 2020

Question 1

(1)

According to previous study we know that the $\hat{\beta}_{OLS}$ is derived as:

$$\hat{\beta}_{OLS} = \frac{\sum_{t=1}^{T}(X_t - \bar{X})(y_t - \bar{y})}{\sum_{t=1}^{T}(X_t - \bar{X})^2} = \frac{\sum_{t=1}^{T}(X_t - \bar{X})y_t}{\sum_{t=1}^{T}(X_t - \bar{X})^2} = \sum_{t=1}^{n} \omega_t y_t = \vec{\omega} \cdot \vec{y}$$

where $\omega_t = \dfrac{X_t - \bar{X}}{\sum_{t=1}^{T}(X_t - \bar{X})^2}$

if we view $\hat{\beta}_{OLS} = f(\vec{y})$, we can easily prove that:

$$f(\alpha\vec{y}) = \alpha f(\vec{y}) \ \ and \ \ f(\vec{y} + \vec{z}) = f(\vec{y}) + f(\vec{z})$$

which means that $\hat{\beta}_{OLS}$ is a linear estimator

(2)

$\hat{\beta}_{OLS}$ is unbiased estimator means $E(\hat{\beta}_{OLS}) = \beta$, first recall that:

$$\begin{aligned}\hat{\beta}_{OLS} &= \sum_{t=1}^{T}\omega_t y_t = \sum_{t=1}^{T}\omega_t(\alpha + \beta X_t + u_t) \\ &= \alpha\sum_{t=1}^{T}\omega_t + \beta\sum_{t=1}^{n}\omega_t X_t + \sum_{t=1}^{T}\omega_t u_t \\ &= \beta + \sum_{t=1}^{T}\omega_t u_t\end{aligned}$$

then we take expectation of $\hat{\beta}_{OLS}$:

$$E(\hat{\beta}_{OLS}) = E(\beta + \sum_{t=1}^{T}\omega_t u_t) = \beta + \sum_{t=1}^{T}\omega_t E(u_t) = \beta \qquad q.e.d$$

(3)

Suppose that $\tilde{\beta}$ is any unbiased linear estimator, which can be written as:

$$\begin{aligned}\tilde{\beta} &= \sum_{t=1}^{T}c_t y_t = \sum_{t=1}^{T}(\omega_t + d_t)y_t \\ &= \sum_{t=1}^{T}(\omega_t + d_t)(\alpha + \beta X_t + u_t) \\ &= \alpha\underbrace{\sum_{t=1}^{T}\omega_t}_{=0} + \beta\underbrace{\sum_{t=1}^{T}\omega_t X_t}_{=1} + \sum_{t=1}^{T}\omega_t u_t + \alpha\sum_{t=1}^{T}d_t + \beta\sum_{t=1}^{T}d_t X_t + \sum_{t=1}^{T}d_t u_t \\ &= \beta + \alpha\sum_{t=1}^{T}d_t + \beta\sum_{t=1}^{T}d_t X_t + \sum_{t=1}^{T}\omega_t u_t + \sum_{t=1}^{T}d_t u_t\end{aligned}$$

Then we take expectation of $\tilde{\beta}$, notice that $E(u_t) = 0$

$$E(\tilde{\beta}) = \beta + \alpha\sum_{t=1}^{T}d_t + \beta\sum_{t=1}^{T}d_t x_t + \sum_{t=1}^{T}\omega_t\underbrace{E(u_t)}_{=0} + \sum_{t=1}^{T}d_t\underbrace{E(u_t)}_{=0}$$

$$E(\tilde{\beta}) = \beta + \alpha \sum_{t=1}^{T} d_t + \beta \sum_{t=1}^{T} d_t x_t = \beta$$
$$i.e. \sum_{t=1}^{T} d_t = 0 \ and \ \sum_{t=1}^{T} d_t x_t = 0 \tag{1}$$

Next we take the variance of $\tilde{\beta}$, notice that $V(u_t) = \sigma^2$.

$$V(\tilde{\beta}) = V(\beta + \sum_{t=1}^{T}(\omega_t + d_t)u_t) = V(\sum_{t=1}^{T}(\omega_t + d_t)u_t) = \sum_{t=1}^{T} V((\omega_t + d_t)u_t)$$
$$= \sum_{t=1}^{T}(\omega_t + d_t)^2 V(u_t)$$
$$= \sigma^2(\sum_{t=1}^{T} \omega_t^2 + \sum_{t=1}^{T} \omega_t d_t + \sum_{t=1}^{T} d_t^2)$$

According to result (1) we know that:

$$\sum_{t=1}^{T} \omega_t d_t = \frac{\sum_{t=1}^{T}(X_t - \bar{X})d_t}{\sum_{t=1}^{T}(X_t - \bar{X})^2} = \frac{\overbrace{\sum_{t=1}^{T} d_t X_t}^{=0} - \bar{X} \overbrace{\sum_{t=1}^{T} d_t}^{=0}}{\sum_{t=1}^{T}(X_t - \bar{X})^2} = 0$$

then we can rewrite $V(\tilde{\beta})$ as:

$$V(\tilde{\beta}) = \sigma^2(\sum_{t=1}^{T} \omega_t^2 + \sum_{t=1}^{T} d_t^2)$$
$$= \sigma^2 \sum_{t=1}^{T} \omega_t^2 + \sigma^2 \sum_{t=1}^{T} d_t^2$$
$$= V(\hat{\beta}_{OLS}) + \sigma^2 \sum_{t=1}^{T} d_t^2 \qquad \geq V(\hat{\beta}_{OLS})$$

Thus we have proved the efficiency of OLS estimator

(4)

according to weak law of large number(WLLN):

$$\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})(u_t - \bar{u}) \xrightarrow{p} Cov(X_t, u_t) = 0$$

P.S :

$$Cov(X_t, u_t) = E(X_t u_t) - E(X_t)\underbrace{E(u_t)}_{=0} = E(E(X_t u_t|X_t)) = E(X_t \underbrace{E(u_t|X_t)}_{=0}) = 0$$

we also assume that:

$$\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2 \xrightarrow{p} m < \infty$$

Then we look at $\hat{\beta}_{OLS}$:

$$\hat{\beta}_{OLS} = \beta + \sum_{t=1}^{T}\omega_t u_t = \beta + \frac{\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})(u_t - \bar{u})}{\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2}$$

$$as \quad T \longrightarrow \infty, \quad \frac{\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})(u_t - \bar{u})}{\frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2} \longrightarrow \frac{0}{m}$$

thus we have:

$$\hat{\beta}_{OLS} \longrightarrow \beta \quad as \quad n \longrightarrow \infty \quad q.e.d$$

(5)

Following the Center Limit theorem(CLT):

$$\frac{\sum_{t=1}^{T}\omega_t u_t - E(\sum_{t=1}^{T}\omega_t u_t)}{\sqrt{V(\sum_{t=1}^{T}\omega_t u_t)}} = \frac{\sum_{t=1}^{T}\omega_t u_t - \sum_{t=1}^{T}\omega_t \overbrace{E(u_t)}^{=0}}{\sqrt{V(\sum_{t=1}^{T}\omega_t u_t)}}$$
$$= \frac{\sum_{t=1}^{T}\omega_t u_t}{\sigma\sqrt{\sum_{t=1}^{T}\omega_t^2}}$$
$$= \frac{\hat{\beta}-\beta}{\sigma/\sqrt{\sum_{t=1}^{T}(X_t-\bar{X})^2}}$$
$$= \frac{\sqrt{T}\hat{\beta}-\beta}{\sigma/\sqrt{\frac{1}{T}\sum_{t=1}^{T}(X_t-\bar{X})^2}} \longrightarrow N(0,1)$$

where $\sum_{t=1}^{T}\omega_t u_t = \hat{\beta}-\beta$ and $\sum_{t=1}^{T}\omega_t^2 = \dfrac{1}{\sum_{t=1}^{T}(X_t-\bar{X})^2}$

as $T \longrightarrow \infty$ we substitute $\frac{1}{T}\sum_{t=1}^{T}(X_t-\bar{X})$ with its convergence value m:

$$\frac{\sqrt{T}(\hat{\beta}-\beta)}{\sigma/\sqrt{m}} \longrightarrow N(0,1)$$

$$\sqrt{T}(\hat{\beta}-\beta) \longrightarrow N(0,\frac{\sigma^2}{m})$$