Econometrics I: Solutions of Homework 3

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1 Solutions

1.1 Question 1 (1)

The OLS estimator of β is

$$\hat{\beta} = \frac{\sum_t (y_t - \bar{y})(X_t - \bar{X})}{\sum_t (X_t - \bar{X})^2} \tag{1}$$

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Then, substituting $y_t = \alpha + \beta X_t + u_t$ into this equation yields

$$\begin{aligned}
\hat{\beta} &= \frac{\sum_t (\alpha + \beta X_t + u_t - \bar{y})(X_t - \bar{X})}{\sum_t (X_t - \bar{X})^2} \\
&= \frac{\beta \sum_t (X_t - \bar{X}) X_t + \sum_t (X_t - \bar{X}) u_t}{\sum_t (X_t - \bar{X})^2} \\
&= \beta + \frac{\sum_t (X_t - \bar{X}) u_t}{\sum_t (X_t - \bar{X})^2} \\
&= \beta + \sum_t \omega_t u_t
\end{aligned}$$

The moment-generating function of $\hat{\beta}$ is defined by

$$M(\theta) \equiv E[\exp(\theta(\beta + \sum_t \omega_t u_t))] \tag{2}$$

Then, we have

$$\begin{aligned}
M(\theta) &= E[\exp(\theta\beta + \theta \sum_t \omega_t u_t)] \\
&= E[\exp(\theta\beta) \exp(\theta \sum_t \omega_t u_t))] \\
&= \exp(\theta\beta) E[\exp(\theta \sum_t \omega_t u_t))] \\
&= \exp(\theta\beta) \prod_{t=1}^{T} E[\exp(\theta \omega_t u_t)] \qquad (3)
\end{aligned}$$

Since $u_t \sim N(0, \sigma^2)$,

$$E[\exp(\theta \omega_t u_t)] = M(\theta \omega_t) = \exp(\sigma^2 (\theta \omega_t)^2 / 2).$$

Substituting this into (3) yields

$$\begin{aligned}
M(\theta) &= \exp(\theta\beta) \prod_t \exp(\sigma^2 (\theta \omega_t)^2 / 2) \\
&= \exp(\theta\beta + \frac{1}{2} \sigma^2 \theta^2 \sum_t \omega_t^2)
\end{aligned}$$

This implies that the exact distribution of $\hat{\beta}$ is

$$\hat{\beta} \sim N(\beta, \sigma^2 \sum_t \omega^2) \tag{4}$$

We will derive the exact distribution more formally, using the following properties:

$$M^{(k)}(0) = E(X^k) \tag{5}$$

where $M^{(k)}(\theta) = \partial^k M(\theta)/\partial\theta^k$. $E(X^k)$ is called *k-th moment* of random variable X. First, we will calculate first and second-order derivative of MGF as follows:

$$\begin{aligned}
M^{(1)}(\theta) &= (\beta + \sigma^2\theta \sum_t \omega_t^2) \exp(\cdot), \\
M^{(2)}(\theta) &= (\sigma^2 \sum_t \omega_t^2) \exp(\cdot) + (\beta + \sigma^2\theta \sum_t \omega_t^2)^2 \exp(\cdot).
\end{aligned}$$

By evaluating two equations at $\theta = 0$, we obtain first and second moment:

$$\begin{aligned}
E(\hat{\beta}) &= \beta \\
E(\hat{\beta}^2) &= \sigma^2 \sum_t \omega_t^2 + \beta^2.
\end{aligned}$$

Finally, variance of $\hat{\beta}$ is

$$V(\hat{\beta}) = E[(\hat{\beta} - E(\hat{\beta}))^2] = E(\hat{\beta}^2) - E(\hat{\beta})^2 = \sigma^2 \sum_t \omega_t^2.$$

1.2 Question 1 (2)

Let $Z \equiv (\hat{\beta} - \beta)/\sigma\sqrt{\sum_t \omega_t^2}$. Then, $Z \sim N(0,1)$ since $\hat{\beta}$ is normally distributed, and

$$\begin{aligned}
E(Z) &= \frac{E(\hat{\beta}) - \beta}{\sigma\sqrt{\sum_t \omega_t^2}} = 0 \\
V(Z) &= \frac{E(\hat{\beta}^2) - \beta^2}{\sigma^2 \sum_t \omega_t^2} = 1
\end{aligned}$$

Using a property that $ks^2/\sigma^2 \sim \chi^2(k)$ where k is a degree of freedom and s^2 is unbiased and consistent estimator of σ^2, we obtain

$$(T-2)\frac{s^2}{\sigma^2} = (T-2)\frac{s^2 \sum_t \omega_t^2}{\sigma^2 \sum_t \omega_t^2} \sim \chi^2(T-2).$$

Let $V \equiv (T-2)s^2 \sum_t \omega_t^2/\sigma^2 \sum_t \omega_t^2$. Then,

$$\frac{Z}{\sqrt{V/(T-2)}} = \frac{\hat{\beta} - \beta}{\sigma\sqrt{\sum_t \omega_t^2}} \Bigg/ \sqrt{\frac{s^2 \sum_t \omega_t^2}{\sigma^2 \sum_t \omega_t^2}} = \frac{\hat{\beta} - \beta}{s\sqrt{\sum_t \omega_t^2}} \sim t(T-2).$$

Thus, $(\hat{\beta} - \beta)/s\sqrt{\sum_t \omega_t^2}$ is a t-distribution with $T - 2$ degrees of freedom.

2 Review: Distributions

2.1 Normal Distribution

Let $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$.

- **Property 1**. $aX + b \sim N(a\mu + b, a^2\sigma^2)$
- **Property 2 (standarization)**. $Z = (X - \mu)/\sigma \sim N(0, 1)$
- **Property 3 (Reproduction)**. Suppose that X and Y are mutually independent. Then, $X+Y \sim N(\mu + \mu_Y, \sigma^2 + \sigma_Y^2)$.

Proof. First, prove Property 1.

$$\begin{aligned} E[\exp(\theta(aX + b))] =& E[\exp(\theta a X)] \exp(\theta b) \\ =& \exp(\mu\theta a + \sigma^2(\theta a)^2/2) \exp(\theta b) \\ =& \exp(\theta(a\mu + b) + \theta^2(\sigma a)^2/2) \end{aligned}$$

Note that second equality comes from $X \sim N(\mu, \sigma^2)$ and its MGF $M(\theta) = \exp(\mu\theta + \sigma^2\theta^2/2)$. This implies that $aX + b \sim N(a\mu + b, a^2\sigma^2)$.
Second, prove Property 2.

$$\begin{aligned} E[\exp(\theta(X - \mu)/\sigma)] =& E[\exp((\theta/\sigma)X)]/\exp(\mu(\theta/\sigma)) \\ =& \exp(\mu(\theta/\sigma) + \sigma^2(\theta/\sigma)^2/2)/\exp(\mu(\theta/\sigma)) \\ =& \exp(\theta^2/2) \end{aligned}$$

This is equivalent the moment-generating function whose random variable is a standard normal distribution.
Third, prove Property 3. To prove it, we first show that the moment-generating function of $X + Y$ is $M_X(t)M_Y(t)$.

$$M_{X+Y}(t) = E[\exp(t(X + Y))] = E[\exp(tX)\exp(tY)] = E[\exp(tX)]E[\exp(tY)] = M_X(t)M_Y(t).$$

Third equality holds since $E(XY) = E(X)E(Y)$ only if X and Y are independent. Finally, we obtain

$$
\begin{aligned}
M_{X+Y}(\theta) &= \exp(\mu\theta + \theta^2\sigma^2/2)\exp(\mu_Y\theta + \theta^2\sigma_Y^2/2) \\
&= \exp(\theta(\mu + \mu_Y) + \theta^2(\sigma^2 + \sigma_Y^2)/2).
\end{aligned}
$$

This implies that $X + Y \sim N(\mu + \mu_Y, \sigma^2 + \sigma_Y^2)$.

2.2 Chi-squared Distribution

Let $X \sim N(\mu, \sigma^2)$. Consider $Z = (X - \mu)/\sigma \sim N(0, 1)$. Then,

$$
V = \sum_i^n z_i^2 \sim \chi^2(n).
$$

If μ are unknown parameter, and we use sample mean $\bar{x} = \sum_i x_i/n$ instead of μ, then

$$
V = \sum_i^n \hat{z}_i^2 \sim \chi^2(n-1),
$$

where $\hat{z}_i \equiv (x_i - \bar{x})/\sigma$.

Recall that unbiased estimator of σ^2 is $s^2 = \sum_i (x_i - \bar{x})^2/(n-1)$. Then, we obtain

$$
\begin{aligned}
\sigma^2 V &= (n-1)s^2 \\
V &= (n-1)\frac{s^2}{\sigma^2}
\end{aligned}
$$

2.3 t Distribution

Let $X \sim N(\mu, \sigma^2)$. The sample means is $\bar{x} = \sum_i x_i/n$, and the unbiased sample variance is $s^2 = \sum_i (x_i - \bar{x})^2/(n-1)$. Then,

$$
T = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t(n-1)
$$

Moreover,

$$
T = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \bigg/ \frac{s/\sqrt{n}}{\sigma/\sqrt{n}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \bigg/ \frac{s}{\sigma} = \frac{Z_{\bar{x}}}{\sqrt{V/(n-1)}} \sim t(n-1),
$$

where $Z_{\bar{x}} \sim N(0, 1)$ by CLT.

Generally, $Z \sim N(0, 1)$, $V \sim \chi^2(k)$, and Z is independent of V. Then, $Z/\sqrt{V/k} \sim \chi^2(k)$.