Econometrics I: Solutions of the homework #4

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Contents

1 Question solution

(1)

In multiple regression case, the regression function can be written as:

$$\underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_T nn \end{pmatrix}}_{\in\mathbb{R}^T} = \underbrace{\begin{pmatrix} x_{1,1} & \cdots & x_{1,k} \\ \vdots & \ddots & \vdots \\ x_{1,T} & \cdots & x_{T,k} \end{pmatrix}}_{\in\mathbb{M}_(T\times k)(\mathbb{R})} \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}}_{\in\mathbb{R}^k} + \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_T \end{pmatrix}}_{\in\mathbb{R}^T} \iff y = X\beta + u$$

*If you have any errors in handouts and materials, please contact me via lvang12@hotmail.com

Then we replace β by $\hat{\beta}$ and we can have the following expression:

$$y = X\hat{\beta} + e$$

where $\hat{\beta}$ is the OLS estimator and e is $n \times 1$ vector of residual

Next we can write the sum of squared residuals as follows:

$$S(\hat{\beta}) = \sum_{t=1}^{n} e_i^2 = e'e = (y - X\hat{\beta})'(y - X\hat{\beta}) = (y - \hat{\beta}'y')(y - X\hat{\beta})$$

$$= y'y - y'X'\hat{\beta} - \hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} = y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$

To minimize $S(\hat{\beta})$ with respect to $\hat{\beta}$, we set the first derivative of $S(\hat{\beta})$ equal to zero:

$$\frac{\partial S(\hat{\beta})}{\partial \hat{\beta}} = -2X'y + 2X'X\hat{\beta} = 0$$

Notice here we applied the matrix calculus rules:

$$\frac{\partial a'x}{\partial x} = a \qquad \frac{\partial x'Ax}{\partial x} = 2Ax$$

Solve the equation we can obtain the OLS estimator as:

$$\hat{\beta} = (X'X)^{-1}X'y$$

Noticing that in order to satisfy the multivariate minimum condition, the second order derivative:

$$\frac{\partial^2 S(\hat{\beta})}{\partial \hat{\beta} \partial \hat{\beta}'} = 2X'X$$

has to be a positive define matrix

(2)

In order to calculate the mean and variance of $\hat{\beta}$ we first need to rewrite the $\hat{\beta}$ as:

$$\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u$$

$$= \beta + (X'X)^{-1}X'u$$

Then we take the conditional expectation for both sides:

$$E(\hat{\beta}|X) = E(\beta + (X'X)^{-1}X'u|X) = \beta + (X'X)^{-1}X'E(u|X) = \beta$$

where $E(u|X) = 0$ by the Gauss-Markov theorem

$$\begin{aligned} V(\hat{\beta}|X) &= E((\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X) = E((X'X)^{-1}X'u((X'X)^{-1}X'u)') \\ &= E((X'X)^{-1}X'uu'X(X'X)^{-1}|X) \\ &= (X'X)^{-1}X'E(uu'|X)X(X'X)^{-1} \\ &= (X'X)^{-1}X'\sigma^2 I_T X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1} \end{aligned}$$

(3)

Notice the moment-generating function of $X \sim N(u, \Sigma)$ is given by:

$$\phi(\theta) \equiv E(exp(\theta' X)) = E(exp(\theta u + \frac{1}{2}\theta' \Sigma \theta)$$

In our case we know that the standard error $u \sim N(0, \sigma^2 I_T)$. i.e.

$$\phi(\theta_u) \equiv E(exp(\theta_u' X)) = E(exp(\frac{1}{2}\theta_u' \theta_u)$$

Next in order to derive a distribution of $\hat{\beta}$, we write the moment-generating function of $\hat{\beta}$ as follows:

$$\begin{aligned}\phi(\theta) &\equiv E(exp(\theta_\beta' \hat{\beta})) = E(exp(\theta_\beta' \beta + \theta_\beta' (X'X)^{-1} X' u) \\ &= exp(\theta_\beta' \beta) E(exp(\theta_\beta' (X'X)^{-1} X' u)) = exp(\theta_\beta' \beta) \phi_u(\theta_\beta' (X'X)^{-1} X') \\ &= exp(\theta_\beta' \beta) exp(\frac{\sigma^2}{2}\theta_\beta' (X'X)^{-1} \theta_\beta) = exp(\theta_\beta' \beta + \frac{\sigma^2}{2}\theta_\beta' (X'X)^{-1} \theta_\beta)\end{aligned}$$

where $\theta_u = X(X'X)^{-1}\theta_\beta$

This indicate that:

$$\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$$

2 Extension

moment-generating function

MGF is a useful tool in terms of calculating the moment of a random variable, first we know that:

$$E(X)\ is\ the\ 1^{th}\ moment$$
$$E(X^2)\ is\ the\ 2^{th}\ moment$$
$$\vdots$$
$$E(X^k)\ is\ the\ k^{th}\ moment$$

In order to calculate the moment in a simple way, we define the Moment Generating Function as:

$$\phi_x(\theta) = E(e^{\theta X}) = \begin{cases} \sum\limits_{all\ x} e^{\theta x} p(x) \quad discrete \\ \\ \int\limits_{-\infty}^{+\infty} e^{\theta x} f(x) dx \quad continuous \end{cases}$$

then we can obtain the nth moment by calculating the nth order derivative and evaluate by $\theta = 0$:

$$E(X^n) = \frac{d^n}{d\theta^n} \phi_x(\theta) \bigg|_{\theta=0}$$

proof

the exponential function e^x can be expanded as:

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} \cdots$$

therefore:

$$e^{\theta X} = 1 + \theta X + \frac{\theta^2}{2!} X^2 + \frac{\theta^3}{3!} X^3 \cdots$$

then we take the expectation for both sides :

$$E(e^{\theta X}) = 1 + \theta E(X) + \frac{\theta^2}{2!} E(X^2) + \frac{\theta^3}{3!} E(X^3) \cdots$$

Next take the derivative with respect to θ and take the value at $\theta = 0$

$$\frac{d}{d\theta}E(e^{\theta X})\Big|_{\theta=0} = 0 + E(X) + \theta E(X^2)\cdots\Big|_{\theta=0} = E(X)$$

Likewise, if we were to calculate the nth moment, we can simply find it by taking the nth order derivative and evaluate by $\theta = 0$

$$\frac{d^n}{d\theta^n}E(e^{\theta X})\Big|_{\theta=0} = 0 + \cdots + 0 + E(X^n) + \theta E(X^{n+1})\cdots\Big|_{\theta=0} = E(X^n)$$