Econometrics I: Solutions of the homework #6

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1 Question 1

1.1 (1)

In order to prove:

$$\frac{(\hat{\beta}-\beta)'X'X(\hat{\beta-\beta})}{\sigma^2} \sim \chi^2(k)$$

*If you have any errors in handouts and materials, please contact me via lvang12@hotmail.com

First recall the OLS estimator $\hat{\beta} = \beta + (X'X)^{-1}Xu$. Substitute $\hat{\beta} - \beta$ with $(X'X)^{-1}Xu$ then the original expression becomes:

$$(\hat{\beta} - \beta)'X'X(\hat{\beta - \beta}) = ((X'X)^{-1}X'u)'X'X(X'X)^{-1}X'u$$

$$= u'X(X'X)^{-1}X'X(X'X)^{-1}X'u = u'X(X'X)^{-1}X'u$$

Notice

if $X \sim N(0, I_k)$ then $X'AX \sim \chi^2(tr(A))$ where A is symmetric and idempotent, i.e., $A'A = A$

In our case X refers to $\frac{u}{\sigma}$, A refers to $X(X'X)^{-1}X$, then we can derive:

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(tr(X(X'X)^{-1}X'))$$

According to trace property we know that $tr(ABC) = tr(BCA)$

$$tr(X(X'X)^{-1}X') = tr((X'X^{-1}X'X)) = tr(I_k) = k$$

Thus we proved that:

$$\frac{(\hat{\beta} - \beta)'X'X(\hat{\beta - \beta})}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

1.2 (2)

If random variables X and Y both follow normal distribution and $Cov(X, Y) = 0$, that indicates X and Y are independent of each other.

Notice

$Cov(X,Y) = 0 \iff Cov(g(X), h(Y)) = 0$

In our question $s^2 = f(e) = \frac{1}{T-k}e'e$. Therefore, to prove $\hat{\beta}$ is independent of s^2 we only need to prove $Cov(\hat{\beta}, e) = 0$

Proof:

$$Cov(e,\hat{\beta}) = E(e(\hat{\beta}-\beta)') = E((I_T - X(X'X)^{-1}X')u((X'X)^{-1}X'u)')$$

$$= E((I_T - X(X'X)^{-1}X')uu'X(X'X)^{-1}) = (I_T - X(X'X)^{-1}X')E(uu')X(X'X)^{-1}$$

$$= (I_T - X(X'X)^{-1}X')(\sigma^2 I_T)X(X'X)^{-1} = \sigma^2(I_T - X(X'X)^{-1}X')X(X'X)^{-1}$$

$$= \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}X'X(X'X)^{-1}) = \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}) = 0$$

where $e = M_x u = (I_T - P_x)u = (I_n - X(X'X)^{-1}X')u$

1.3 (3)

Notice

$\frac{U/n}{V/m} \sim F(n,m)$ when $U \sim \chi^2(n)$, $V \sim \chi^2(m)$, U is independent of V

Previously we already proved that:

$$\frac{(\hat{\beta}-\beta)'X'X(\beta \hat{-} \beta)}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

$$\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\sigma^2} \sim \chi^2(T-k)$$

Thus we can obtain:

$$\frac{\frac{(\hat{\beta}-\beta)'X'X(\hat{\beta-\beta})}{\sigma^2}/k}{\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\sigma^2}/(T-k)} \sim F(k, T-k)$$

1.4 (4)

In order to prove this question we first rewrite the sum as the vector form:

$$\sum_{i=1}^{T}(y_i-\bar{y})^2 = (y-\bar{y})'(y-\bar{y})$$

where:

$$(y-\bar{y}) = \begin{pmatrix} y_1-\bar{y} \\ y_2-\bar{y} \\ \vdots \\ y_T-\bar{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} - \begin{pmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix} = y - \frac{1}{T}ii'y = (I_T - \frac{1}{T}ii')y$$

Therefore $(y-\bar{y})'(y-\bar{y})$ can be written as:

$$(y-\bar{y})'(y-\bar{y}) = y'(I_T - \frac{1}{T}ii')(I_T - \frac{1}{T}ii')y = y'(I_T - \frac{1}{T}ii')y$$

Here $I_T - \frac{1}{T}ii'$ is a symmetric and idempotent matrix and we will prove this property in the next question.

1.5 (5)

Notice that $i = (1, \cdots, 1)'$, thus we can write ii' as:

$$ii' = \begin{pmatrix} 1_{1,1} & \cdots & 1_{1,T} \\ \vdots & \ddots & \vdots \\ 1_{1,T} & \cdots & 1_{T,T} \end{pmatrix}$$

then

$$I_T - \frac{1}{n}ii' = \begin{pmatrix} 1-\frac{1}{T} & -\frac{1}{T} & \cdots - \frac{1}{T} \\ \vdots & \ddots & \vdots \\ -\frac{1}{T} & \cdots & 1-\frac{1}{T} \end{pmatrix}$$

Apparently this is a symmetric matrix i.e. $I_T - \frac{1}{n}ii' = (I_T - \frac{1}{n}ii')^T$

Next we prove idempotent property by showing $(I_T - \frac{1}{T}ii')(I_T - \frac{1}{T}ii') = I_T - \frac{1}{T}ii'$

First we expand the expression as:

$$(I_T - \frac{1}{T}ii')(I_T - \frac{1}{T}ii') = I_T - \frac{1}{T}ii' - \frac{1}{T}ii' + \frac{1}{T^2}ii'ii' \tag{1}$$

where

$$ii'ii' = \begin{pmatrix} 1_{1,1} & \cdots & 1_{1,T} \\ \vdots & \ddots & \vdots \\ 1_{1,T} & \cdots & 1_{T,T} \end{pmatrix} \begin{pmatrix} 1_{1,1} & \cdots & 1_{1,T} \\ \vdots & \ddots & \vdots \\ 1_{1,T} & \cdots & 1_{T,T} \end{pmatrix} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,T} \\ \vdots & \ddots & \vdots \\ T_{1,T} & \cdots & T_{T,T} \end{pmatrix} = Tii'$$

Substitude $ii'ii'$ with Tii' in (1) and we can obtain:

$$(I_T - \frac{1}{T}ii')(I_T - \frac{1}{T}ii') = I_T - \frac{1}{T}ii'$$

This proves that $I_T - \frac{1}{T}ii'$ is a idempotent matrix

2 Question 2

2.1 (6)

Expand the numerator we can obtain:

$$\frac{(X-ui)'(X-ui)}{\sigma^2} = \frac{\sum_{i=1}^{n}(X-u)^2}{\sigma^2} = \sum_{i=1}^{n}(\frac{X-u}{\sigma})^2$$

Obviously $\frac{X-u}{\sigma} \sim N(0,1)$, thus the above expression is the sum of nth squared standard normal distribution. i.e.

$$\frac{(X-ui)'(X-ui)}{\sigma^2} \sim \chi^2(n)$$

2.2 (7)

In the previous Q1(4) we have proved that:

$$(I_n - \frac{1}{n}ii')X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} = X - \bar{X}$$

In the same manner:

$$(I_n - \frac{1}{n}ii')ui = u(I_n - \frac{1}{n}ii')i = u(i-\bar{i}) = 0$$

where u is a scalar and $i = \bar{i} = (1, \cdots, 1)'$

Next applying this result to $(I_n - \frac{1}{n}ii')(X-ui)$, we can obtain:

$$(I_n - \frac{1}{n}ii')(X-ui) = (I_n - \frac{1}{n}ii')X - (I_n - \frac{1}{n}ii')ui = (X - \bar{X})$$

Thus we proved that:

$$
\begin{aligned}
(X-ui)'(I_n-\frac{1}{n}ii')(X-ui) &= (X-ui)'(I_n-\frac{1}{n}ii')(I_n-\frac{1}{n}ii')(X-ui) \\
&= (X-\bar{X})'(X-\bar{X}) = \sum_{i=1}^{n}(X-\bar{X})^2
\end{aligned}
$$

2.3 (8)

Applying the same method in Q1(1), we can obtain:

$$\frac{(X-ui)'(I_n-\frac{1}{n}ii')(X-ui)}{\sigma^2} \sim \chi^2(tr(I_n-\frac{1}{n}ii'))$$

where $tr(I_n-\frac{1}{n}ii') = tr(I_n) - tr(\frac{1}{n}ii') = n-1$

Thus:

$$\frac{(X-ui)'(I_n-\frac{1}{n}ii')(X-ui)}{\sigma^2} \sim \chi^2(n-1)$$