Econometrics I: Solutions of Homework 7

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May 29, 2020

Contents

1 Solutions

1.1 (1)

A transformation matrix M is defined by

$$M = I_T - \frac{1}{T} ii',$$

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where I_T is $T \times T$ identity matrix [1]. Then,

$$Mi = I_T i - \frac{1}{T} i(i'i) = i - \frac{1}{T} iT = 0,$$

and

$$Me = I_T e - \frac{1}{T} i(i'e) = e - \frac{1}{T} i \sum_{t=1}^{T} e_t = e.$$

Note that third equality comes from the property of the OLS estimator, that is, $\sum_t e_t/T = \bar{e} = 0$.

1.2 (2)

First, I will show the first equality. By $Me = e$, premultiplying e on M yields

$$\begin{aligned} Me &= My - MX\hat{\beta} \\ e &= My - MX\hat{\beta}. \end{aligned}$$

Then, we obtain

$$e'e = y'M'My - \hat{\beta}'X'M'MX\hat{\beta} = y'My - \hat{\beta}X'MX\hat{\beta}$$

The second equality comes from the fact that M is symmetric and idempotent. The M is idempotent because

$$\begin{aligned} M^2 &= \left(I_T - \frac{1}{T} ii'\right)\left(I_T - \frac{1}{T} ii'\right) \\ &= I_T - \frac{1}{T} ii' - \frac{1}{T} ii' + \frac{1}{T^2} i(i'i)i' \\ &= I_T - \frac{1}{T} ii' - \frac{1}{T} ii' + \frac{1}{T^2} i(T)i' \\ &= I_T - \frac{1}{T} ii' = M. \end{aligned}$$

Second, I will show the second equality. By $Me = e$ and $Mi = 0$, premultiplying $e = y - i\hat{\beta}_1 - X_2\hat{\beta}_2$ on M gives

$$\begin{aligned} Me &= My - Mi\hat{\beta}_1 - MX_2\hat{\beta}_2 \\ e &= My - MX_2\hat{\beta}_2 \end{aligned}$$

[1] This matrix is different from the matrix M that I used in solutions to HW5.

Then, we obtain

$$
\begin{aligned}
e'e &= (My - MX_2\hat{\beta}_2)'(My - MX_2\hat{\beta}_2) \\
&= y'My - \hat{\beta}_2'X_2'My - y'MX_2\hat{\beta}_2 + \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 \\
&= y'My - \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 - \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 + \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 \\
&= y'My - \hat{\beta}_2'X_2'MX_2\hat{\beta}_2.
\end{aligned}
$$

The third equality comes from $X_2'My = X_2'MX_2\hat{\beta}_2$. This holds because

$$
\begin{aligned}
X_2'e &= X_2'My - X_2'MX_2\hat{\beta}_2 \\
0 &= X_2'My - X_2'MX_2\hat{\beta}_2
\end{aligned}
$$

Note that $X_2'e = 0$ holds since $X'e = X'y - X'X\hat{\beta} = (X'X)\hat{\beta} - X'X\hat{\beta} = 0$ by $\hat{\beta} = (X'X)^{-1}X'y$.

1.3 (3)

Since $y'My$ is a scalar,

$$
R^2 = 1 - \frac{e'e}{y'My} = \frac{y'My}{y'My} - \frac{e'e}{y'My} = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My}
$$

1.4 (4)

$$
R\hat{\beta} = R(X'X)^{-1}X'y = R(X'X)^{-1}X'(X\beta + u) = R\beta + R(X'X)^{-1}X'u
$$

Since u is normally distributed, Rb is also normally distributed. Expectation and variance of Rb are as follows:

$$
\begin{aligned}
&E(R\hat{\beta}) = R\beta \\
&V(R\hat{\beta}) = E[(Rb - R\beta)(Rb - R\beta)'] = E[R(X'X)^{-1}X'uu'X'(X'X)^{-1}R'] = \sigma^2 R(X'X)^{-1}R'
\end{aligned}
$$

Thus, the distribution of Rb is

$$
R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R').
$$

1.5 (5)

By the question (4), we can replace $R\beta$ by r if the null hypothesis is correct. Thus,

$$R\hat{\beta} \sim N(r, \sigma^2 R(X'X)^{-1}R'),$$

or

$$(R\hat{\beta} - r) \sim N(0, \sigma^2 R(X'X)^{-1}R'),$$

1.6 (6)

$$R = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = (0, I_{k-1})$$

where R is $(k-1) \times k$ matrix. Thus, $G = k - 1$ and $r = 0$.

1.7 (7)

We will show that, given R and r,

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2.$$

By the solution to (6), define $R = (0, I_{k-1})$ and $r = 0$. Then, $R\hat{\beta} - r = \hat{\beta}_2$.

Next, given R and r as defined above, we will show $(R(X'X)^{-1}R')^{-1} = X_2' M X_2$. First, by $X = (i, X_2)$,

$$(X'X)^{-1} = \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} \begin{pmatrix} i & X_2 \end{pmatrix} \right)^{-1} = \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where B_{ij} is unknown matrices. Then, we have

$$R(X'X)^{-1}R' = \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = \begin{pmatrix} B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = B_{22}$$

Thus, we only need to calculate B_{22}. By the property of the inverse of a partitioned matrix,

$$\begin{aligned} B_{22} &= (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \\ &= (X_2'I_TX_2 - X_2'(\frac{1}{T}ii')X_2)^{-1} \\ &= (X_2'(I_T - \frac{1}{T}ii')X_2)^{-1} \\ &= (X_2'MX_2)^{-1} \end{aligned}$$

Hence, $(R(X'X)^{-1}R')^{-1} = ((X_2'MX_2)^{-1})^{-1} = X_2'MX_2$.

Finally, given $R = (0, I_{k-1})$ and $r = 0$, we obtain

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2.$$

1.8 (8)

By solutions to (6) and (7), test statistic for $H_0 : \beta_2 = 0$ is

$$\frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(T-k)} \sim F(k-1, T-k)$$

We will show that

$$\frac{R^2/(k-1)}{(1-R^2)/(T-k)} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(T-k)},$$

where R^2 is the coefficient of determination.

By solutions to (3), we obtain

$$(y'My)R^2 = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2,$$

and,

$$(y'My)(1-R^2) = e'e.$$

Since $y'My$ is scalar, we can obtain

$$\frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2/(k-1)}{e'e/(T-k)} = \frac{R^2/(k-1)}{(1-R^2)/(T-k)}$$

1.9 (9)

First, we test $\beta = 0$, using t-statistic. Recall that t-statistic is given by

$$t = \frac{\hat{\beta} - \beta}{s/\sqrt{\sum_t (X_t - \bar{X})^2}} \sim t(T-k)$$

where s^2 is unbiased and consistent estimator of σ^2. Since $V(\hat{\beta}) = \sigma^2/\sum_t (X_t - \bar{X})^2$, we can calculate t-statistic as follows:

$$t = \frac{\hat{\beta} - \beta}{SE(\hat{\beta})}.$$

Thus,

$$t = \frac{0.65 - 0}{0.240} = 2.70833$$

Under the degree of freedom is $4 - 2 = 2$, the test statistic at 1%, 5%, and 10% significance level is 9.9248, 4.3072, and 2.9200, respectively. Thus, we cannot reject the null hypothesis $\beta = 0$.

Second, we test $\beta = 0$, using F-statistic with R^2. By solutions to (8), a test statistic is given by

$$\frac{R^2/(k-1)}{(1-R^2)/(T-k)} = \frac{0.786/(2-1)}{(1-0.786)/(4-2)} = 7.345794$$

Under $F \sim F(1,2)$, the test statistic at 1%, 5%, and 10% significance level is 98.50251, 18.51282, and 8.526316, respectively. Thus, we cannot reject the null hypothesis $\beta = 0$.

Overall, F-test obtains the same result as t-test. Note that the square of t-statistic is approximate to F-statistic, that is, $t^2 = (2.70833)^2 = 7.335069 \approx 7.345794$.