Econometrics I: Solutions of the homework #8

LU ANG*

June 10, 2020

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1 Question 1

1.1 (1)

We apply the Lagrange multiplier to calculate the restricted estimator. In order to minimize $(y - X\beta)'(y - X\beta)$ with the restriction $R\beta = r$. We can write the Loss function as:

$$L = (y - X\tilde{\beta})'(y - X\tilde{\beta}) - 2\tilde{\lambda}'(R\tilde{\beta} - r)$$

Here $\tilde{\beta}$ and $\tilde{\lambda}$ are the estimators that minimize L. Then the F.O.C can be obtained:

$$\frac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0$$
$$\frac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0$$

Solving the equation for $\tilde{\beta}$ we have following expression:

$$\tilde{\beta} = (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda} \tag{1}$$

*If you have any errors in handouts and materials, please contact me via lvang12@hotmail.com

Multiply R by both side we have following expression:

$$R\tilde{\beta} = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}$$

Because we have the restriction $R\tilde{\beta} = r$, we substitute the Left side:

$$r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}$$

Thus we can solve $\tilde{\lambda}$ as:

$$\tilde{\lambda} = (R(X'X)^{-1}R')^{-1}(r - R\hat{\beta})$$

Next we substitute $\tilde{\lambda}$ back into equation (1) we can obtain:

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\hat{\beta})$$

1.2 (2)

In the previous section we have learned that:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(T - k)} \sim F(G, T - k)$$

Where $G = Rank(R)$

Using $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\hat{\beta})$, we can derive that:

$$(X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta} - \tilde{\beta}$$

Multiply R by both sides we can have the following expression:

$$(R\hat{\beta} - r) = R(\hat{\beta} - \tilde{\beta})$$

Substitute this expression back into the numerator we can obtain:

$$
\begin{aligned}
(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r) &= (\hat{\beta}-\tilde{\beta})'R'(R(X'X)^{-1}R')^{-1}R(\hat{\beta}-\tilde{\beta}) \\
&= (\hat{\beta}-\tilde{\beta})'X'X(\hat{\beta}-\tilde{\beta})
\end{aligned}
$$

Moreover the numerator is represented as follows:

$$
\begin{aligned}
(y-X\tilde{\beta})'(y-X\tilde{\beta}) &= (y-X\hat{\beta}-X(\tilde{\beta}-\hat{\beta}))'(y-X\hat{\beta}-X(\tilde{\beta}-\hat{\beta})) \\
&= (y-X\hat{\beta})'(y-X\hat{\beta}) + (\tilde{\beta}-\hat{\beta})'X'X(\tilde{\beta}-\hat{\beta}) \\
&\quad -(y-X\hat{\beta})'X(\tilde{\beta}-\hat{\beta}) - (\tilde{\beta}-\hat{\beta})'X'(y-X\hat{\beta}) \\
&= (y-X\hat{\beta})'(y-X\hat{\beta}) + (\hat{\beta}-\tilde{\beta})'X'X(\hat{\beta}-\tilde{\beta})
\end{aligned}
$$

where $X'(y-X\hat{\beta}) = X'\hat{u} = 0$

Summarizing, we have following representation:

$$
\begin{aligned}
(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r) &= (\hat{\beta}-\tilde{\beta})'X'X(\hat{\beta}-\tilde{\beta}) \\
&= (y-X\tilde{\beta})'(y-X\tilde{\beta}) - (y-X\hat{\beta})'(y-X\hat{\beta}) \\
&= \tilde{u}'\tilde{u} - \hat{u}'\hat{u}
\end{aligned}
$$

Therefore we have:

$$
\frac{(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)/G}{(y-X\hat{\beta})'(y-X\hat{\beta})/(T-k)} = \frac{(\tilde{u}'\tilde{u}-\hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(G, T-k)
$$

2 Question 2

2.1 (3)

We have the hypothesis as:

$$
H_0 : \alpha_1 = \alpha_2 = 0
$$

$$H_1 : \alpha_1 \neq 0 \ or \ \alpha_2 \neq 0$$

The matrix form $R\beta = r$, where:

$$\beta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice in production function (2) does not contain the variables K_t and L_t, Therefore it satisfy our H_0, so that we can take production function (1) as our original regression and production function (2) as our restricted regression. i.e. $G = Rank(R) = 2$ and $T - k = (1997 - 1969) - 3 = 25$

Thus we have:

$$F = \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(2, 25)$$

where $\tilde{u}$ is the residual from product function (2) and $\hat{u}$ is the residual from product function (1). Recall that $\hat{u}'\hat{u} = \hat{\sigma}^2_{(1)}(T - k_{(1)})$ and $\tilde{u}'\tilde{u} = \hat{\sigma}^2_{(2)}(T - k_{(2)})$

Then the F scores is calculated as:

$$F = \frac{(0.00354801 \times 26 - 0.00141869 \times 25)/2}{0.00141869} = 20.0117$$

Under $F \sim F(2, 25)$, the test statistic at 1%, 5%, and 10% significance level are 5.57, 3.39, and 2.53, respectively. All are smaller than our F score. Thus, we should reject the null hypothesis.

2.2 (4)

In order to test whether the production function is homogeneous, we have the hypothesis as:

$$H_0 : \alpha_1 + \alpha_2 = 1$$
$$H_0 : \alpha_1 + \alpha_2 \neq 1$$

The matrix form $R\beta = r$, where:

$$\beta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \end{pmatrix}$$

Notice we can rewrite production function (2) as:

$$\begin{aligned} log(Y_t/L_t) &= \beta_0 + \beta_1 log(K_t/L_t) + u_t \\ logY_t - logL_t &= \beta_0 + \beta_1 logK_t - \beta_1 logL_t + u_t \\ logY_t &= \beta_0 + \beta_1 logK_t + (1-\beta_1) logL_t + u_t \end{aligned}$$

In this case if we view β_1 as α_1 and $(1-\beta_1)$ as α_2. it satisfy our $H_0 : \alpha_1 + \alpha_2 = 1$. Therefore we can take product function (1) as our original regression and product function (2) as our restricted regression. i.e. $G = Rank(R) = 1$, $T - k = 28 - 3 = 25$

Thus we have:

$$F = \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(1, 25)$$

Then the F scores is calculated as:

$$F = \frac{(0.00354801 \times 26 - 0.00141869 \times 25)}{0.00141869} = 40.0235$$

Under $F \sim F(1, 25)$, the test statistic at 1%, 5%, and 10% significance level are 7.77, 4.24, and 2.92, respectively. All are smaller than our F score. Thus, we should reject the null hypothesis.

2.3 (5)

In order to test whether there are structural changes, we have the hypothesis as:

$$H_0 : \gamma_3 = \gamma_4 = \gamma_5 = 0$$

$$H_1 : \gamma_3 \neq 0 \;\; or \;\; \gamma_4 \neq 0 \;\; or \;\; \gamma_5 \neq 0$$

the matrix form $R\beta = r$, where:

$$\beta = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case we consider product function (3) as our original regression and product function (1) as our restricted regression. i.e. $G = Rank(R) = 3$, $T - k = 28 - 6 = 22$

Thus we have:

$$F = \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(3, 22)$$

Where $\tilde{u}$ is the residual from product function (1) and $\hat{u}$ is the residual from product function (3). Then the F scores is calculated as:

$$F = \frac{(0.00141869 \times 25 - 0.00145010 \times 22)/3}{0.00145010} = 0.8194$$

Under $F \sim F(3, 22)$, the test statistic at 1%, 5%, and 10% significance level are 4.82, 3.05, and 2.35, respectively. All are greater than our F score. Thus, we cannot reject the null hypothesis.