Econometrics I: Solutions of Homework #9

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1 Solutions

1.1 (1)

Using the matrix of eigenvectors of Ω (denoted by A) and the diagonal matrix Λ whose elements are eigenvalues λ_i, the matrix Ω can be diagonalized as follows:

$$A'\Omega A = \Lambda,$$

that is,

$$\Omega = A\Lambda A' \tag{1}$$

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Since Ω is a positive definite matrix, all its eigenvalues are positive. Thus, Λ is factored into

$$\Lambda = \Lambda^{1/2}\Lambda^{1/2}$$

where $\Lambda^{1/2} = diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Substituting in eq (1) gives

$$\Lambda = A\Lambda^{1/2}\Lambda^{1/2}A' = (A\Lambda^{1/2})(A\Lambda^{1/2})'.$$

Thus, we obtain

$$P = A\Lambda^{1/2}.$$

1.2 (2)

Since $\{u_t\}_{t=1}^T$ are mutually independent, $Cov(u_t, u_s) = 0$ for $t \neq s$. Thus, the variance-covariance matrix is given by

$$E(uu') = \sigma^2\Omega = \sigma^2 \begin{pmatrix} z_1^2 & 0 & \cdots & 0 \\ 0 & z_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & z_T^2 \end{pmatrix}$$

1.3 (3)

Let us introduce the lag operator L such that $L^s x_t = x_{t-s}$ for $s \geq 1$. Then, a first-order autoregressive scheme, $u_t = \rho u_{t-1} + \epsilon_t$, can be rewritten as

$$\begin{aligned}(1 - \rho L)u_t &= \epsilon_t \\ u_t &= (1 - \rho L)^{-1}\epsilon_t.\end{aligned}$$

Without loss of generality, assume $|\rho| < 1$. Then, the inverse of first lag operator results in a series of infinite differences, that is,

$$\begin{aligned}u_t &= (1 + \rho L + \rho^2 L^2 + \cdots)\epsilon_t \\ &= \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots .\end{aligned}$$

Thus, $E(u_t) = 0$ and the second-order moment of u_t is

$$\begin{aligned}
E(u_t^2) &= E[\epsilon_t^2 + \epsilon_t(\rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots) + \rho^2\epsilon_{t-1}^2 + \rho\epsilon_{t-1}(\epsilon_t + \rho^2\epsilon_{t-2} + \cdots) + \cdots] \\
&= E[\epsilon_t^2 + \rho^2\epsilon_{t-1}^2 + \rho^4\epsilon_{t-2}^2 + \cdots] \\
&= (1 + \rho^2 + \rho^4 + \cdots)\sigma^2 = \frac{1}{1-\rho}\sigma^2
\end{aligned}$$

Note that $E(\epsilon_t\epsilon_{t-s}) = 0$ since $\{\epsilon_t\}_t$ are mutually independent. Also, it is simple to establish that

$$\begin{aligned}
E(u_t u_{t-s}) &= E[\epsilon_t(\epsilon_{t-s} + \rho\epsilon_{t-s-1} + \cdots) + \cdots + \rho^s\epsilon_{t-s}(\epsilon_{t-s} + \rho\epsilon_{t-s-1} + \cdots) + \cdots] \\
&= E(\rho^s\epsilon_{t-s}^2) = \rho^s\frac{\sigma^2}{1-\rho^2}
\end{aligned}$$

This leads to $V(u_t) = (1-\rho^2)^{-1}\sigma^2$ and $Cov(u_t, u_{t-s}) = \rho^s(1-\rho^2)^{-1}\sigma^2$. Finally, the variance-covariance matrix is given by

$$E(uu') = \sigma^2\Omega = \sigma^2\frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{pmatrix}$$

1.4 (4)

Let A be a $T \times T$ nonsingular transformation matrix. Then, we premultiply the regression model by A to obtain

$$Ay = (AX)\beta + Au. \tag{2}$$

Then, the variance-covariance matrix of u_t is

$$E(uu') = E(Auu'A') = \sigma^2 A\Omega A'.$$

If it were possible to specify A such that $A\Omega A' = I_T$, then we could apply OLS to the transformed variables Ay and AX, and the estimates would have all the optimal properties of OLS (i.e. BLUE estimator).

Using the property which we proved in the question (1), we can find the matrix A which will hold $A\Omega A' = I_T$. Since Ω is a positive definite matrix, there exists a nonsingular matrix P such that

$\Omega = PP'$. Since P is nonsingular, $P^{-1}\Omega P'^{-1} = I_T$. The appropriate matrix A is given by

$$A = P^{-1}.$$

Applying OLS to the transformed regression model (2) then gives

$$\begin{aligned} b &= (X'A'AX)^{-1}X'A'Ay \\ &= (X'P'^{-1}P^{-1}X)^{-1}X'P'^{-1}P^{-1}y \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y \end{aligned}$$

1.5 (5)

Using the original regression model, we obtain the OLS estimator, $\hat{\beta} = (X'X)^{-1}X'y$. Then, we have

$$\begin{aligned} E(\hat{\beta}) &= E(\beta + (X'X)^{-1}X'u) = \beta, \\ V(\hat{\beta}) &= E((X'X)^{-1}X'uu'X(X'X)^{-1}) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}. \end{aligned}$$

The variance of GLS estimator which we derive in the question (4) is given by

$$\begin{aligned} V(b) &= E((X'\Omega^{-1}X)^{-1}X'\Omega^{-1}uu'\Omega^{-1}X(X'\Omega^{-1}X)^{-1}) \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}. \end{aligned}$$

Then,

$$\begin{aligned} & V(\hat{\beta}) - V(b) \\ =& \sigma^2[(X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}]\Omega[(X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}]' \\ =& \sigma^2 A\Omega A' \end{aligned}$$

Ω is a variance-covariance matrix, which is a positive definite matrix. This implies that $A\Omega A'$ is also a positive definite matrix (proof can be found at footnote 1 in the solution key #5). Hence, $V(\hat{\beta}) - V(b)$ is a positive definite matrix, which implies that the GLS estimator is more efficient than the OLS estimator if the error term is not homoscedasticity.