Econometrics I: Solutions of the homework #10

LU ANG*

June 23, 2020

Contents

1 Question 1

1.1 (1)

The likelihood function is defined as $L(\theta; X) = \prod_{i=1}^{n} f(X_i; \theta)$ in our case:

$$L(\theta; X) = (2\pi\sigma^2)^{-n/2} exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - u)^2)$$

The maximum likelihood estimate of θ is the θ such that:

$$\max_{\theta} L(\theta; X) \iff \max_{\theta} logL(\theta; X)$$

*If you have any errors in handouts and materials, please contact me via lvang12@hotmail.com

Therefore we can take log for both sides and the log likelihood function can be written as:

$$logL(\theta; X) = -\frac{n}{2}log(2\pi) - \frac{n}{2}log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - u)^2$$

1.2 (2)

Obtain the first order condition:

$$\frac{\partial logL(\theta; X)}{\partial u} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - u) = 0$$
$$\frac{\partial logL(\theta; X)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - u)^2 = 0$$

Solving the equation we can obtain the MLE estimators of u and σ as:

$$\hat{u} = \frac{1}{n}\sum_{i=1}^{n}X_i$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{u})^2$$

1.3 (3)

the variance of MLE estimators are calculated as:

$$V(\hat{u}) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{\sigma^2}{n}$$

$$\begin{aligned} V(\hat{\sigma}^2) &= V(\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{u})^2) \\ &= V(\frac{\sigma^2}{n\sigma^2}\sum_{i=1}^{n}(X_i - \hat{u})^2) \\ &= \frac{\sigma^4}{n^2}V(\sum_{i=1}^{n}(\frac{X_i - \hat{u}}{\sigma})^2) \end{aligned}$$

Recall that $\sum_{i=1}^{n}(\frac{X_i-\hat{u}}{\sigma})^2 \sim \chi^2(n-1)$ which has the variance $2(n-1)$ therefore:

$$V(\hat{u}) = \frac{2\sigma^4(n-1)}{n^2}$$

1.4 (4)

Fisher's information matrix $I(\theta)$ is given as follows:

$$
\begin{aligned}
I(\theta) = -E(\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}) &= -E\begin{pmatrix} \frac{\partial^2 logL(\theta;X)}{\partial u^2} & \frac{\partial^2 logL(\theta;X)}{\partial u \partial\sigma^2} \\ \frac{\partial^2 logL(\theta;X)}{\partial\sigma^2 \partial u} & \frac{\partial^2 logL(\theta;X)}{\partial(\sigma^2)^2} \end{pmatrix} \\
&= -E\begin{pmatrix} -\frac{n}{\sigma^2} & -\frac{n}{\sigma^4}\sum_{i=1}^{n}(X_i - u) \\ -\frac{n}{\sigma^4}\sum_{i=1}^{n}(X_i - u) & \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(X_i - u)^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}
\end{aligned}
$$

1.5 (5)

Compare (3) and (4) we can find when n is large the variance of MLE will approximate to $I(\theta)^{-1}$. i.e.

$$
\begin{pmatrix} \hat{u} \\ \hat{\sigma}^2 \end{pmatrix} \sim N\left(\begin{pmatrix} u \\ \sigma^2 \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}\right)
$$