Econometrics I: Solutions of Homework #11

Hiroki Kato *

July 3, 2020

Contents

1 Solutions

1.1 (1)

Since we assume u_t is normally distributed with $E(u_t) = 0$ and $V(u_t) = \sigma^2$, the density function of u_t is given by

$$f(u_t) = \frac{1}{(2\pi\sigma^2)^{(1/2)}} \exp\left(-\frac{1}{2\sigma^2}u_t^2\right).$$

By the mutual independent assumption, the joint density function of $u_1, \ldots, u_T$ is given by

$$f(u_1, \ldots, u_T) = \frac{1}{(2\pi\sigma^2)^{(T/2)}} \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{T} u_t^2\right)$$

*e-mail: vge008kh@student.econ.osaka-u.ac.jp. Room 503. If you find any errors in handouts and materials, please contact me via e-mail.

Transforming u_t, we have the following likelihood function;[1]

$$L(\alpha, \beta, \sigma^2 | y_1, \dots, y_T) \equiv \frac{1}{(2\pi\sigma^2)^{(T/2)}} \exp\left(-\frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - \alpha - \beta x_t)^2 \right). \quad (1)$$

1.2 (2)

To obtain the log-likelihood function, we take a natural logarithm of (1) as follows:

$$\log L(\alpha, \beta, \sigma^2 | y_1, \dots, y_T) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - \alpha - \beta x_t)^2$$

Taking a first-order derivative with respect to unknown parameters $(\alpha, \beta, \sigma^2)$ yields

$$\frac{\partial L(\alpha, \beta, \sigma^2 | y_1, \dots, y_T)}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{t=1}^{T} (y_t - \alpha - \beta x_t) = 0 \quad (2)$$

$$\frac{\partial L(\alpha, \beta, \sigma^2 | y_1, \dots, y_T)}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^{T} (y_t - \alpha - \beta x_t) x_t = 0 \quad (3)$$

$$\frac{\partial L(\alpha, \beta, \sigma^2 | y_1, \dots, y_T)}{\partial \sigma^2} = -\frac{T}{2} \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=1}^{T} (y_t - \alpha - \beta x_t)^2 = 0 \quad (4)$$

Rewriting the equation (2) and the equation (3) yields

$$T\bar{y} - \tilde{\alpha} T - \tilde{\beta} T \bar{x} = 0,$$
$$\sum_t y_t x_t - \tilde{\alpha} T \bar{x} - \tilde{\beta} \sum_t x_t^2 = 0,$$

where $\bar{y} = \sum_t y_t / T$ and $\bar{x} = \sum_t x_t / T$.

First, using the first equation, we can obtain the ML estimator of α, denoted by $\tilde{\alpha}$, as follows;

$$\tilde{\alpha} = \bar{y} - \tilde{\beta} \bar{x}.$$

Substituting this estimator into the second equation, we obtain the ML estimator of β, denoted by

[1] We implicitly assume that x_t is non-stochastic fixed variable.

$\tilde{\beta}$, as follows;

$$-\tilde{\beta}[\sum_t x_t^2 - T\bar{x}^2] + \sum_t y_t x_t - T\bar{x}\bar{y} = 0$$
$$\tilde{\beta} = \frac{\sum_t y_t x_t - T\bar{x}\bar{y}}{\sum_t x_t^2 - T\bar{x}^2}$$
$$\tilde{\beta} = \frac{\sum_t (y_t - \bar{y})(x_t - \bar{x})}{\sum_t (x_t - \bar{x})^2}$$

Solving the equation (4) gives the ML estimator of σ^2, denoted by $\tilde{\sigma}^2$, as follows;

$$\tilde{\sigma}^2 = \frac{\sum_{t=1}^T (y_t - \tilde{\alpha} - \tilde{\beta}x_t)^2}{T}.$$

In summary, the ML estimator of θ is

$$\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2)' = \left(\bar{y} - \tilde{\beta}\bar{x}, \frac{\sum_t (y_t - \bar{y})(x_t - \bar{x})}{\sum_t (x_t - \bar{x})^2}, \frac{\sum_{t=1}^T (y_t - \tilde{\alpha} - \tilde{\beta}x_t)^2}{T}\right)'.$$

1.3 (3)

By the equation (2), (3), and (4),

$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial \alpha^2} = -\frac{T}{\sigma^2}$$
$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial \beta^2} = -\frac{\sum_t x_t^2}{\sigma^2}$$
$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial (\sigma^2)^2} = \frac{T}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{t=1}^T (y_t - \alpha - \beta x_t)^2 = \frac{T}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{t=1}^T u_t^2$$
$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial \alpha \partial \beta} = -\frac{\sum_t x_t}{\sigma^2}$$
$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial \alpha \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{t=1}^T (y_t - \alpha - \beta x_t) = -\frac{1}{\sigma^4}\sum_{t=1}^T u_t$$
$$\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial \beta \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{t=1}^T (y_t - \alpha - \beta x_t)x_t = -\frac{1}{\sigma^4}\sum_{t=1}^T u_t x_t$$

By the distributional assumption, we have $\sum_t E(u_t) = 0$, $\sum_t E(u_t)x_t = 0$, and $\sum_t E(u_t^2) = \sum_t [E(u_t^2) - E(u_t)] = \sum_t V(u_t) = T\sigma^2$. Thus, the information matrix $I(\theta)$ is given by

$$I(\theta) = -E\left(\frac{\partial^2 L(\alpha, \beta, \sigma^2 | y_1, \ldots, y_T)}{\partial\theta\partial\theta'}\right) = \begin{pmatrix} \frac{T}{\sigma^2} & \frac{\sum_t x_t}{\sigma^2} & 0 \\ \frac{\sum_t x_t}{\sigma^2} & \frac{\sum_t x_t^2}{\sigma^2} & 0 \\ 0 & 0 & \frac{T}{2\sigma^4} \end{pmatrix}$$

1.4 (4)

Step 1: Variance-Covariance Matrix of MLE

By the Cramer-Rao lower bound theorem, the inverse of information matrix $I(\theta)^{-1}$ provides a lower bound of the variance-covariance matrix for unbiased estimators of θ.[2] Then, the variance-covariance matrix of $\tilde{\theta}$ is

$$\begin{pmatrix} \frac{\sigma^2}{T} \frac{\sum_t x_t^2}{\sum_t (x_t - \bar{x})^2} & -\sigma^2 \frac{\bar{x}}{\sum_t (x_t - \bar{x})^2} & 0 \\ -\sigma^2 \frac{\bar{x}}{\sum_t (x_t - \bar{x})^2} & \frac{\sigma^2}{\sum_t (x_t - \bar{x})^2} & 0 \\ 0 & 0 & \frac{2\sigma^4}{T} \end{pmatrix}$$

Step 2: Derive the expectation and variance of OLSE

The OLS estimator of β, denoted by $\hat{\beta}$, is equivalent to the ML estimator of β, denoted by $\tilde{\beta}$. That is,

$$\hat{\beta} = \frac{\sum_t (y_t - \bar{y})(x_t - \bar{x})}{\sum_t (x_t - \bar{x})^2}.$$

Thus, we have $E(\hat{\beta}) = \beta$ and $V(\hat{\beta}) = \sigma^2 / (\sum_t (x_t - \bar{x})^2)$. This implies that there is no difference between the MLE and the OLSE of β.

The OLS estimator of σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{T-2} \sum_{t=1}^{T} (y_t - \tilde{\alpha} - \tilde{\beta} x_t)^2.$$

[2] Try to derive by yourself

This estimator is unbiased: $E(\hat{\sigma}^2) = \sigma^2$. Since the OLSE is different from the ML estimator of β, $\tilde{\beta}$. the ML estimator is biased estimator of σ^2, that is,

$$E(\tilde{\sigma}^2) = \frac{T-2}{T} E(\hat{\sigma^2}) = \frac{T-2}{T} \sigma^2.$$