Econometrics I: Solutions of the homework #12

LU ANG*

July 7, 2020

Contents

1 Question 1

1.1 (1)

In order to Obtain the variance matrix of $y = (y_1, y_2, \cdots, y_t)$ we first rewrite y_t as:

*If you have any errors in handouts and materials, please contact me via lvang12@hotmail.com

$$
\begin{aligned}
y_t &= \rho y_{t-1} + \epsilon_t \\
&= \rho^2 y_{t-2} + \epsilon_t + \rho\epsilon_{t-1} \\
&\vdots \\
&= \rho^\tau y_{t-\tau} + \epsilon_t + \rho\epsilon_{t-1} + \cdots + \rho^{\tau-1}\epsilon_{t-\tau+1} \\
&\vdots \\
&= \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots
\end{aligned}
$$

where $\rho^\tau y_{t-\tau}$ goes to 0 when $|\rho| < 1$ as τ goes to infinity

Then we can obtain variance as:

$$
\begin{aligned}
V(y_t) &= V(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots) \\
&= V(\epsilon_t) + \rho^2 V(\epsilon_{t-1}) + \rho^4 V(\epsilon_{t-2}) + \cdots \\
&= \sigma^2(1 + \rho^2 + \rho^4 + \cdots) \\
&= \frac{\sigma^2}{1-\rho^2}
\end{aligned}
$$

Next denote the covariance of y_t and $y_{t-\tau}$ as $\gamma(\tau)$:

$$
\begin{aligned}
\gamma(\tau) &= Cov(y_t, y_{t-\tau}) \\
&= E(y_t y_{t-\tau}) = E\left((\rho^\tau y_{t-\tau} + \epsilon_t + \rho\epsilon_{t-1} + \cdots + \rho^{\tau-1}\epsilon_{t-\tau+1})y_{t-\tau}\right) \\
&= \rho^\tau E(y_{t-\tau}^2) + E(u_t y_{t-\tau}) + \rho E(u_{t-1}y_{t-\tau}) + \rho^2 E(u_{t-2}y_{t-\tau}) + \cdots \rho^{\tau-1}E(u_{t-\tau+1}y_{t-\tau}) \\
&= \rho^\tau E(y_{t-\tau}^2) \\
&= \rho^\tau \gamma(0)
\end{aligned}
$$

Notice $V(y_t) = \gamma(0)$

Going back to the vector form

$$
E(y) = E\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

$$V(y) = E(yy') = E(\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_T \end{pmatrix})$$

$$= \begin{pmatrix} \gamma(0) & \gamma(1) & & \cdots & \gamma(T-1) \\ \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(T-2) \\ & \gamma(1) & \gamma(0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \gamma(1) \\ \gamma(T-1) & \gamma(T-2) & \cdots & \gamma(1) & \gamma(0) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(0) & \rho\gamma(0) & & \cdots & \rho^{T-1}\gamma(0) \\ \rho\gamma(0) & \gamma(0) & \rho\gamma(0) & \cdots & \rho^{T-2}\gamma(0) \\ & \rho\gamma(0) & \gamma(0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho\gamma(0) \\ \rho^{T-1}\gamma(0) & \rho^{T-2} & \cdots & \rho\gamma(0) & \gamma(0) \end{pmatrix}$$

$$= \gamma(0) \begin{pmatrix} 1 & \rho & & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ & \rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ \rho^{T-1} & \rho^{T-2} & \cdots & \rho & 1 \end{pmatrix}$$

$$= \frac{\sigma^2}{1-\rho^2} \begin{pmatrix} 1 & \rho & & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ & \rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ \rho^{T-1} & \rho^{T-2} & \cdots & \rho & 1 \end{pmatrix} = \Omega$$

1.2 (2)

Based on (1) we know that $y \sim N(0, \Omega)$. Thus, the joint distribution of $y = (y_1, y_2, \cdots, y_T)$ whihch is also the likelihood function can be obtained as:

$$f(y) = (2\pi)^{-T/2}|\Omega|^{-1/2}exp(-\frac{1}{2}y'\Omega^{-1}y)$$

1.3 (3)

Unconditional Mean:

$$\begin{aligned} E(y_t) &= E(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots) \\ &= E(\epsilon_t) + \rho E(\epsilon_{t-1}) + \rho^2 E(\epsilon_{t-2}) + \cdots \\ &= 0 \end{aligned}$$

Unconditional Variance:

$$\begin{aligned} V(y_t) &= V(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots) \\ &= V(\epsilon_t) + \rho^2 V(\epsilon_{t-1}) + \rho^4 V(\epsilon_{t-2}) + \cdots \\ &= \sigma^2(1 + \rho^2 + \rho^4 + \cdots) \\ &= \frac{\sigma^2}{1-\rho^2} \end{aligned}$$

Conditional mean:

$$E(y_t|y_{t-1}, \cdots, y_1) = \rho y_{t-1} + E(\epsilon_t) = \rho y_{t-1}$$

Conditional variance:

$$V(y_t|y_{t-1}, \cdots, y_1) = V(\epsilon_t) = \sigma^2$$

1.4 (4)

From (3) we can know that the unconditional distribution of y_t is given by:

$$f(y_t) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\rho^2)}} exp(-\frac{y_t^2}{2\sigma^2/(1-\rho^2)})$$

the conditional distribution of y_t is given by:

$$f(y_t|y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(y_t - \rho y_{t-1})^2}{2\sigma^2})$$

The innovation form of the likelihood function can be written as:

$$\begin{aligned} f(y_t, y_{t-1}, \cdots, y_1) &= f(y_t|y_{t-1}, \cdots, y_1) f(y_{t-1}, y_{t-2}, \cdots, y_1) \\ &= f(y_t|y_{t-1}, \cdots, y_1) f(y_{t-1}|y_{t-2}, \cdots, y_1) f(y_{t-2}, y_{t-3}, \cdots, y_1) \\ &\vdots \\ &= f(y_t|y_{t-1}, \cdots, y_1) f(y_{t-1}|y_{t-2}, \cdots, y_1) f(y_{t-2}, y_{t-3}, \cdots, y_1) \cdots f(y_2|y_1) f(y_1) \\ &= f(y_1) \prod_{t=2}^{T} f(y_t|y_{y-1}, \cdots, y_1) \\ &= \frac{1}{\sqrt{2\pi\sigma^2/(1-\rho^2)}} exp(-\frac{y_1^2}{2\sigma^2/(1-\rho^2)}) \times \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(y_t - \rho y_{t-1})^2}{2\sigma^2}) \end{aligned}$$

1.5 (5)

Set P^{-1} such that:

$$P^{-1}\Omega P'^{-1} = I_T$$

i.e.

$$\Omega = PP'$$

We can construct P^{-1} as:

$$P^{-1} = \frac{1}{\sigma} \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \cdots & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix}$$

Thus:

$$P^{-1}y = \frac{1}{\sigma} \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \cdots & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}$$

$$= \frac{1}{\sigma} \begin{pmatrix} \sqrt{1-\rho^2} y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_T - \rho y_{T-1} \end{pmatrix}$$

Then:

$$y'\Omega^{-1}y = y'P'^{-1}P^{-1}y$$

$$= \frac{1}{\sigma^2} \begin{pmatrix} \sqrt{1-\rho^2} y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_T - \rho y_{T-1} \end{pmatrix}^T \begin{pmatrix} \sqrt{1-\rho^2} y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_T - \rho y_{T-1} \end{pmatrix}$$

$$= \frac{1}{\sigma^2} \left((1-\rho^2) y_1^2 + \sum_{t=2}^{T} (y_t - \rho y_{t-1})^2 \right) \qquad (1)$$

On the hand by using the matrix determinant property $det(A^{-1}) = \frac{1}{det(A)}$ and $det(AB) = det(A)det(B)$ we can easily obtain:

$$|\Omega| = \frac{1}{|\Omega^{-1}|} = \frac{1}{|P'^{-1}||P^{-1}|} = \frac{\sigma^{2T}}{1-\rho^2} \tag{2}$$

Substitute expression (1) and (2) back into $f(y) = (2\pi)^{-T/2}|\Omega|^{-1/2}exp(-\frac{1}{2}y'\Omega^{-1}y)$ we can easily find that the likelihood function we obtained in question (2) and (4) are the same.

Finally we can calculate the inverse of P^{-1} to obtain:

$$P = \frac{\sigma}{\sqrt{1-\rho^2}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \rho & \sqrt{1-\rho^2} & 0 & \cdots & 0 \\ \rho^2 & \rho\sqrt{1-\rho^2} & \sqrt{1-\rho^2} & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \rho^{T-1} & \rho^{T-2}\sqrt{1-\rho^2} & \cdots & \rho\sqrt{1-\rho^2} & \sqrt{1-\rho^2} \end{pmatrix}$$