Example: AR(p) model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$.

1. **Variance of AR(p) Process:**

   Under the stationarity condition (i.e., the p solutions of x from $\phi(x) = 0$ are outside the unit circle),

   $$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1 \rho(1) - \cdots - \phi_p \rho(p)}.$$

   Note that $\gamma(\tau) = \rho(\tau)\gamma(0)$.

   Solve the following simultaneous equations for $\tau = 0, 1, \cdots, p$:

   $$\gamma(\tau) = \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau})$$
   $$= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \cdots + \phi_p \gamma(\tau - p), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \cdots + \phi_p \gamma(\tau - p) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}$$

2. **Estimation of AR(p) Model:**

   1. **OLS:**

   $$\min_{\phi_1, \cdots, \phi_p} \sum_{t=p+1}^{T} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p})^2$$

   2. **MLE:**

   $$\max_{\phi_1, \cdots, \phi_p} \log f(y_T, \cdots, y_1)$$

   where

   $$\log f(y_T, \cdots, y_1) = \log f(y_p, \cdots, y_2, y_1) + \sum_{t=p+1}^{T} \log f(y_t | y_{t-1}, \cdots, y_1),$$

   $$f(y_p, \cdots, y_2, y_1) = (2\pi)^{-p/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(y_1 \, y_2 \, \cdots \, y_p) V^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}\right)$$

$$V = \gamma(0) \begin{pmatrix} 1 & \rho(1) & \cdots & \rho(p-2) & \rho(p-1) \\ \rho(1) & 1 & & \rho(p-3) & \rho(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(1) & 1 \end{pmatrix}$$

$$f(y_t | y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p})^2\right)$$

3. Yule=Walker (   ) Equation:

Multiply $y_{t-1}, y_{t-2}, \cdots, y_{t-p}$ on both sides of $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} +$

$\epsilon_t = y_t$, take expectations for each case, and divide by the sample variance $\hat{\gamma}(0)$.

$$
\begin{pmatrix}
1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-2) & \hat{\rho}(p-1) \\
\hat{\rho}(1) & 1 & & \hat{\rho}(p-3) & \hat{\rho}(p-2) \\
\vdots & \vdots & & \vdots & \vdots \\
\hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & \hat{\rho}(1) & 1
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p
\end{pmatrix}
=
\begin{pmatrix}
\hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p)
\end{pmatrix}
$$

where

$$
\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^{T} (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu}), \qquad \hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}.
$$

3. **AR(p) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots \phi_p y_{t-p} + \epsilon_t$

   Mean:

$$
\phi(L)y_t = \mu + \epsilon_t
$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$.

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t$$

Taking the expectation on both sides,

$$E(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) = \phi(1)^{-1}\mu$$
$$= \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

4. **Partial Autocorrelation of AR(p) Process:**

$\phi_{k,k} = 0$ for $k = p + 1, p + 2, \cdots$.

6.4 MA Model

MA (Moving Average   ) Model:

1. MA(q)

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

2. **Invertibility (   ):**

The q solutions of x from $\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q = 0$ are outside the unit circle.

$\implies$ MA(q) model is rewritten as AR(∞) model.

Example: MA(1) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1}$

1. **Mean of MA(1) Process:**

$$E(y_t) = E(\epsilon_t + \theta_1 \epsilon_{t-1}) = E(\epsilon_t) + \theta_1 E(\epsilon_{t-1}) = 0$$

2. **Autocovariance Function of MA(1) Process:**

$$\gamma(0) = E(y_t^2) = E(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = E(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2)$$
$$= E(\epsilon_t^2) + 2\theta_1 E(\epsilon_t \epsilon_{t-1}) + \theta_1^2 E(\epsilon_{t-1}^2) = (1 + \theta_1^2)\sigma_\epsilon^2$$

$$\gamma(1) = \mathrm{E}(y_t y_{t-1}) = \mathrm{E}((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-1} + \theta_1 \epsilon_{t-2})) = \theta_1 \sigma_\epsilon^2$$

$$\gamma(2) = \mathrm{E}(y_t y_{t-2}) = \mathrm{E}((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-2} + \theta_1 \epsilon_{t-3})) = 0$$

3. **Autocorrelation Function of MA(1) Process:**

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \begin{cases} \dfrac{\theta_1}{1 + \theta_1^2}, & \text{for } \tau = 1, \\ 0, & \text{for } \tau = 2, 3, \cdots. \end{cases}$$

Let x be $\rho(1)$.

$$\frac{\theta_1}{1 + \theta_1^2} = x, \qquad \text{i.e.,} \qquad x\theta_1^2 - \theta + x = 0.$$

θ_1 should be a real number.

$$1 - 4x^2 > 0, \qquad \text{i.e.,} \qquad -\frac{1}{2} \leq \rho(1) \leq \frac{1}{2}.$$

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4. **Invertibility Condition of MA(1) Process:**

$$\epsilon_t = -\theta_1 \epsilon_{t-1} + y_t$$

$$= (-\theta_1)^2 \epsilon_{t-2} + y_t + (-\theta_1)y_{t-1}$$

$$= (-\theta_1)^3 \epsilon_{t-3} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2 y_{t-2}$$

$$\vdots$$

$$= (-\theta_1)^s \epsilon_{t-s} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2 y_{t-2} + \cdots + (-\theta_1)^{t-s+1} y_{t-s+1}$$

When $(-\theta_1)^s \epsilon_{t-s} \longrightarrow 0$, the MA(1) model is written as the AR(∞) model, i.e.,

$$y_t = -(-\theta_1)y_{t-1} - (-\theta_1)^2 y_{t-2} - \cdots - (-\theta_1)^{t-s+1} y_{t-s+1} - \cdots + \epsilon_t$$

5. **Likelihood Function of MA(1) Process:**

The autocovariance functions are: $\gamma(0) = (1+\theta_1^2)\sigma_\epsilon^2$, $\gamma(1) = \theta_1\sigma_\epsilon^2$, and $\gamma(\tau) = 0$ for $\tau = 2, 3, \cdots$.

The joint distribution of $y_1, y_2, \cdots, y_T$ is:

$$f(y_1, y_2, \cdots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y'V^{-1}Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \qquad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 & \theta_1 & 0 & \cdots & & 0 \\ \theta_1 & 1 + \theta_1^2 & \theta_1 & \ddots & & \vdots \\ 0 & \theta_1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & 1 + \theta_1^2 & \theta_1 \\ 0 & \cdots & 0 & \theta_1 & 1 + \theta_1^2 \end{pmatrix}.$$

6. **MA(1) +drift:** $y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}$

Mean of MA(1) Process:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L$.

Taking the expectation,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$

Example: MA(2) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$

1. Autocovariance Function of MA(2) Process:

$$\gamma(\tau) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma_\epsilon^2, & \text{for } \tau = 0, \\ (\theta_1 + \theta_1\theta_2)\sigma_\epsilon^2, & \text{for } \tau = 1, \\ \theta_2\sigma_\epsilon^2, & \text{for } \tau = 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. let $-1/\beta_1$ and $-1/\beta_2$ be two solutions of x from $\theta(x) = 0$.

   For invertibility condition, both β_1 and β_2 should be less than one in absolute value.

   Then, the MA(2) model is represented as:

   $$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

$$= (1 + \theta_1 L + \theta_2 L^2)\epsilon_t$$

$$= (1 + \beta_1 L)(1 + \beta_2 L)\epsilon_t$$

AR(∞) representation of the MA(2) model is given by:

$$\epsilon_t = \frac{1}{(1 + \beta_1 L)(1 + \beta_2 L)} y_t$$
$$= \left(\frac{\beta_1/(\beta_1 - \beta_2)}{1 + \beta_1 L} + \frac{-\beta_2/(\beta_1 - \beta_2)}{1 + \beta_2 L} \right) y_t$$

3. **Likelihood Function:**

$$f(y_1, y_2, \cdots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \theta_2 & & 0 \\ \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \ddots & \\ \theta_2 & \theta_1 + \theta_1\theta_2 & \ddots & \ddots & \theta_2 \\ & & \ddots & \ddots & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 \\ 0 & & \theta_2 & \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 \end{pmatrix}$$

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4. **MA(2) +drift:** $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2$.

Therefore,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu$$

Example: MA(q) Model: $\quad y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

1. **Mean of MA(q) Process:**

$$\mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}) = 0$$

2. **Autocovariance Function of MA(q) Process:**

$$\gamma(\tau) = \begin{cases} \sigma_\epsilon^2(\theta_0\theta_\tau + \theta_1\theta_{\tau+1} + \cdots + \theta_{q-\tau}\theta_q) = \sigma_\epsilon^2 \sum_{i=0}^{q-\tau} \theta_i\theta_{\tau+i}, & \tau = 1, 2, \cdots, q, \\ 0, & \tau = q+1, q+2, \cdots, \end{cases}$$

where $\theta_0 = 1$.

3. MA(q) process is stationary.

4. **MA(q) +drift:** $\quad y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$.

Therefore, we have:

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$