

## 6.11 ARCH and GARCH Models

Autoregressive Conditional Heteroskedasticity (ARCH)

Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

### 1. ARCH ( $p$ ) Model

$$\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1 \sim N(0, h_t),$$

where ,

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \dots + \alpha_p \epsilon_{t-p}^2.$$

The unconditional variance of  $\epsilon_t$  is:

$$\sigma_\epsilon^2 = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2 - \dots - \alpha_p}$$

### 2. GARCH ( $p, q$ ) Model

$$\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1 \sim N(0, h_t),$$

where

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 + \beta_1 h_{t-1} + \cdots + \beta_q h_{t-q}.$$

### 3. Application to OLS (Case of ARCH(1) Model):

$$y_t = x_t \beta + \epsilon_t, \quad \epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1 \sim N(0, \alpha_0 + \alpha_1 \epsilon_{t-1}^2).$$

The joint density of  $\epsilon_1, \epsilon_2, \dots, \epsilon_T$  is:

$$\begin{aligned} f(\epsilon_1, \dots, \epsilon_T) &= f(\epsilon_1) \prod_{t=2}^T f(\epsilon_t | \epsilon_{t-1}, \dots, \epsilon_1) \\ &= (2\pi)^{-1/2} \left( \frac{\alpha_0}{1 - \alpha_1} \right)^{-1/2} \exp\left( -\frac{1}{2\alpha_0/(1 - \alpha_1)} \epsilon_1^2 \right) \\ &\quad \times (2\pi)^{-(T-1)/2} \prod_{t=2}^T (\alpha_0 + \alpha_1 \epsilon_{t-1}^2)^{-1/2} \exp\left( -\frac{1}{2} \sum_{t=2}^T \frac{\epsilon_t^2}{\alpha_0 + \alpha_1 \epsilon_{t-1}^2} \right). \end{aligned}$$

The log-likelihood function is:

$$\begin{aligned} \log L(\beta, \alpha_0, \alpha_1; y_1, \dots, y_T) &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log\left(\frac{\alpha_0}{1 - \alpha_1}\right) - \frac{1}{2\alpha_0/(1 - \alpha_1)} (y_1 - x_1\beta)^2 \\ &\quad - \frac{T-1}{2} \log(2\pi) - \frac{1}{2} \sum_{t=2}^T \log(\alpha_0 + \alpha_1(y_{t-1} - x_{t-1}\beta)^2) \\ &\quad - \frac{1}{2} \sum_{t=2}^T \frac{(y_t - x_t\beta)^2}{\alpha_0 + \alpha_1(y_{t-1} - x_{t-1}\beta)^2}. \end{aligned}$$

Obtain  $\alpha_0$ ,  $\alpha_1$  and  $\beta$  such that the log-likelihood function is maximized.

$\alpha_0 > 0$  and  $\alpha_1 > 0$  have to be satisfied.

These two conditions are explicitly included, when the model is modified to:

$$E(\epsilon_t^2 | \epsilon_{t-1}, \epsilon_{t-2}, \dots, \epsilon_1) = \alpha_0^2 + \alpha_1^2 \epsilon_{t-1}^2.$$

### Testing the ARCH(1) Effect:

- (a) Estimate  $y_t = x_t\beta + u_t$  by OLS, and compute  $\hat{\beta}$  and  $\hat{u}_t = y_t - x_t\hat{\beta}$ .
- (b) Estimate  $\hat{u}_t^2 = \alpha_0 + \alpha_1\hat{u}_{t-1}^2$  by OLS. If  $\hat{\alpha}_1$  is significant, there is the ARCH(1) effect in the error term.

This test corresponds to LM test.

### Example: GARCH(1,1) Model

```
. arch sdex l1.sdex l2.sdex, arch(1) garch(1)
```

誤差項の MA 項を GRARCH に

```
(setting optimization to BHHH)
```

```
Iteration 0:   log likelihood = -5089.3558
```

```
Iteration 1:   log likelihood = -5086.7468
```

```
.....  
.....
```

```
Iteration 22:  log likelihood = -5064.9328 (backed up)
```

```
Iteration 23:  log likelihood = -5064.9328
```

```
ARCH family regression
```

Sample: 16 - 516  
 Distribution: Gaussian  
 Log likelihood = -5064.933

Number of obs = 501  
 Wald chi2(2) = 225.19  
 Prob > chi2 = 0.0000

		Coef.	OPG Std. Err.	z	P> z	[95% Conf. Interval]	
sdex							
	sdex						
	L1.	-.6357273	.0426939	-14.89	0.000	-.7194059	-.5520488
	L2.	-.370862	.0466222	-7.95	0.000	-.4622398	-.2794842
	_cons	-55.28043	261.2057	-0.21	0.832	-567.2341	456.6733
ARCH							
	arch						
	L1.	.041632	.0123474	3.37	0.001	.0174317	.0658324
	garch						
	L1.	.9526041	.0148639	64.09	0.000	.9234715	.9817367
	_cons	312143.8	227564.3	1.37	0.170	-133873.9	758161.6

## 7 Vector Autoregressive (VAR) Model – Causality, Impulse Response Function and etc

We can consider VARMA (vector autoregressive moving average) model.

However, it is very difficult to estimate MA terms in the case of vector.

Usually, we consider VAR (Vector Autoregressive) process:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where

$$y_t : k \times 1, \quad \mu : k \times 1, \quad \epsilon_t : k \times 1, \quad \phi_i : k \times k.$$

Rewriting the above equation,

$$\phi(L)y_t = \mu + \epsilon_t,$$

where  $\phi(L) = I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ .

## VAR(1) Model:

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \quad \text{i.e.,} \quad (I_k - \phi_1 L)y_t = \epsilon_t.$$

When  $y_t$  is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L)^{-1} \epsilon_t \\ &= (I_k + \phi_1 L + \phi_1^2 L^2 + \phi_1^3 L^3 + \dots) \epsilon_t \\ &= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \dots \end{aligned}$$

VAR(1)=VMA( $\infty$ )

## VAR(2) Model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \quad \text{i.e.,} \quad (I_k - \phi_1 L - \phi_2 L^2) y_{t-1} = \epsilon_t.$$

When  $y_t$  is stationary, we obtain:

$$\begin{aligned} y_{t-1} &= (I_k - \phi_1 L - \phi_2 L^2)^{-1} \epsilon_t \\ &= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots \end{aligned}$$

VAR(2)=VMA( $\infty$ )



### **VAR( $p$ ) Model:**

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t,$$

i.e.,

$$(I_k - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) y_{t-1} = \epsilon_t.$$

When  $y_t$  is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)^{-1} \epsilon_t \\ &= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots \end{aligned}$$

VAR( $p$ )=VMA( $\infty$ )

## 7.1 Autocovariance Matrix and Autocorrelation Matrix

Let  $y_t$  be a  $k \times 1$  vector.

Autocovariance Function Matrix:

$$\Gamma(\tau) = E((y_t - \mu)(y_{t-\tau} - \mu)'), \quad \tau = 0, 1, 2, \dots,$$

where  $E(y_t) = \mu$ .  $\Gamma(\tau)$  is a  $k \times k$  matrix.

$$\Gamma(\tau) = \Gamma(-\tau)'$$

Autocorrelation Function Matrix:

$$\rho(\tau) = D^{-1/2}\Gamma(\tau)D^{-1/2},$$

where the  $(i, j)$ th element of  $D$  is given by  $\gamma_{ii}(\tau) = V(y_{it})$  for  $i = j$  and zero otherwise.

$$\rho(\tau) = \rho(-\tau)'$$

## 7.2 Granger Causality Test (グレンジャー因果性テスト)

Consider the bivariate case:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & \phi_{12,1} \\ \phi_{21,1} & \phi_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,p} & \phi_{12,p} \\ \phi_{21,p} & \phi_{22,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

Take an example of the first equation.

- Unrestricted Model (Sum of Squared Residuals, denoted by  $SSR_1$ ):

$$\begin{aligned} y_{1,t} &= \mu_1 + \phi_{11,1}y_{1,t-1} + \phi_{12,1}y_{2,t-1} \\ &\quad + \phi_{11,2}y_{1,t-2} + \phi_{12,2}y_{2,t-2} \\ &\quad \dots \\ &\quad + \phi_{11,p}y_{1,t-p} + \phi_{12,p}y_{2,t-p} + \epsilon_{1,t} \\ \longrightarrow \hat{\epsilon}_{1,t} &\longrightarrow \text{Unrestricted VAR}(p) \longrightarrow SSR_1 = \sum_{t=p+1}^T \hat{\epsilon}_{1,t}^2 \end{aligned}$$

Test  $H_0 : \phi_{12,1} = \phi_{12,2} = \dots = \phi_{12,p} = 0$ .

When  $H_0$  is correct, we say there is no causality from  $y_2$  to  $y_1$ .

$\implies$  **Granger Causality Test**

- Restricted Model (Sum of Squared Residuals, denoted by  $SSR_0$ ):

Under  $H_0 : \phi_{12,1} = \phi_{12,2} = \dots = \phi_{12,p} = 0$ , we estimate the following regression:

$$\begin{aligned}y_{1,t} &= \mu_1 + \phi_{11,1}y_{1,t-1} + 0 \times y_{2,t-1} \\ &\quad + \phi_{11,2}y_{1,t-2} + 0 \times y_{2,t-2} \\ &\quad \dots \\ &\quad + \phi_{11,p}y_{1,t-p} + 0 \times y_{2,t-p} + \epsilon_{1,t} \\ &= \mu_1 + \phi_{11,1}y_{1,t-1} \\ &\quad + \phi_{11,2}y_{1,t-2} \\ &\quad \dots\end{aligned}$$

$$+ \phi_{11,p} y_{1,t-p} + \epsilon_1$$

$$\longrightarrow \tilde{\epsilon}_{1,t} \longrightarrow \text{Restricted VAR}(p) \longrightarrow \text{SSR}_0 = \sum_{t=p+1}^T \tilde{\epsilon}_{1,t}^2$$

The number of parameters to be estimated:

- Unrestricted Model:  $2p + 1$   
 $\longrightarrow \text{VAR}(p)$ ,  $p$  lagged coefficients, one constant term  $\longrightarrow 2p + 1$
- Restricted Model:  $p + 1$   
 $\longrightarrow$  The number of restrictions =  $p$  ( $= G$ )

Therefore, asymptotically we have the following distribution:

$$F = \frac{(\text{SSR}_0 - \text{SSR}_1)/p}{\text{SSR}_1/(T - 2p - 1)} \sim F(p, T - 2p - 1),$$

or

$$pF \sim \chi^2(p).$$

In general, we consider testing the Granger causality from  $y_j$  to  $y_i$ .

VAR( $p$ ) model:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where  $y_t : k \times 1$ ,  $\mu : k \times 1$ ,  $\phi_p : k \times k$ ,  $\epsilon_t : k \times 1$ .

$\phi_p$  is given by:

$$\phi_p = \begin{pmatrix} \phi_{11,p} & \cdots & \phi_{1j,p} & \cdots & \phi_{1k,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{i1,p} & \cdots & \phi_{ij,p} & \cdots & \phi_{ik,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{k1,p} & \cdots & \phi_{kj,p} & \cdots & \phi_{kk,p} \end{pmatrix}$$

Test the Granger causality from  $y_j$  to  $y_i$ .

→ Focus on the  $i$ th equation.

$$\begin{aligned}y_{i,t} = & \mu_i + \phi_{i1,1}y_{1,t-1} + \cdots + \phi_{ij,1}y_{j,t-1} + \cdots + \phi_{ik,1}y_{k,t-1} \\ & + \phi_{i1,2}y_{1,t-2} + \cdots + \phi_{ij,2}y_{j,t-2} + \cdots + \phi_{ik,2}y_{k,t-2} \\ & \cdots \\ & + \phi_{i1,p}y_{1,t-p} + \cdots + \phi_{ij,p}y_{j,t-p} + \cdots + \phi_{ik,p}y_{k,t-p} + \epsilon_{i,t}\end{aligned}$$

Suppose that the null hypothesis is:  $H_0 : \phi_{ij,1} = \phi_{ij,2} = \cdots = \phi_{ij,p} = 0$ , while the alternative hypothesis is:  $H_1 : \text{not } H_0$ .

$SSR_0$  = Sum of Squared Residuals under  $H_0$

$SSR_1$  = Sum of Squared Residuals under  $H_1$

Under  $H_0$ , the asymptotic distribution is given by:

$$F = \frac{(\text{SSR}_0 - \text{SSR}_1)/p}{\text{SSR}_1/(T - kp - 1)} \sim F(p, T - kp - 1),$$

or

$$pF \sim \chi^2(p).$$

**Example: VAR( $p$ ):** Data: 1994 年第一四半期 ~ 2014 年第一四半期

gdp = GDP (実質, 10 億円, 季調済, 内閣府 HP から取得)

def = GDP デフレーター (季調済, 内閣府 HP から取得)

r = 貸出約定平均金利 (% , 新規, 総合・国内銀行, 日銀 HP から取得)

m = 通貨流通高 (平均発行高, 億円, 季調済, 日銀 HP から取得)



```

. gen time=_n
. tsset time
    time variable:  time, 1 to 81
                  delta:  1 unit
. gen lgdp=log(gdp)
. gen lm=log(m/(def/100))
. varsoc d.lgdp d.r d.lm

```

time のデータ作成

time が時間データとする

gdp の対数変換

マネーサプライ m を実質化で、対数変換

各変数階差を取って、ラグ次数を決める

Selection-order criteria

Sample: 6 - 81

Number of obs

=

76

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	541.22				1.4e-10	-14.1637	-14.1269	-14.0717
1	571.181	59.923*	9	0.000	8.2e-11*	-14.7153*	-14.5682*	-14.3473*
2	575.715	9.0675	9	0.431	9.2e-11	-14.5978	-14.3404	-13.9537
3	579.55	7.6704	9	0.568	1.1e-10	-14.4619	-14.0942	-13.5418
4	583.767	8.4328	9	0.491	1.2e-10	-14.336	-13.858	-13.1399

Endogenous: D.lgdp D.r D.lm

Exogenous: \_cons

. var d.lgdp d.r d.lm, lags(1)

各変数階差を取って, 3 変数 VAR(1)

Vector autoregression

Sample: 3 - 81  
 Log likelihood = 592.2334  
 FPE = 8.38e-11  
 Det(Sigma\_ml) = 6.18e-11

No. of obs = 79  
 AIC = -14.68945  
 HQIC = -14.54526  
 SBIC = -14.32954

Equation	Parms	RMSE	R-sq	chi2	P>chi2
D_lgdp	4	.010717	0.0422	3.480972	0.3232
D_r	4	.087186	0.2553	27.0782	0.0000
D_lm	4	.009434	0.2903	32.30929	0.0000

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
D_lgdp						
lgdp						
LD.	.2031129	.1119361	1.81	0.070	-.0162778	.4225037
r						
LD.	.0045431	.0120151	0.38	0.705	-.0190061	.0280922
lm						
LD.	.0152162	.1086739	0.14	0.889	-.1977807	.228213
_cons	.0019504	.0019124	1.02	0.308	-.0017978	.0056986

D_r	lgdp							
	LD.	.4341641	.9106374	0.48	0.634	-1.350652	2.218981	
	r							
	LD.	.5085677	.0977469	5.20	0.000	.3169874	.7001481	
lm	LD.	.1845222	.8840978	0.21	0.835	-1.548278	1.917322	
	_cons	-.0202984	.0155578	-1.30	0.192	-.0507912	.0101943	
	-----							
	D_lm	lgdp						
	LD.	-.1972406	.098541	-2.00	0.045	-.3903774	-.0041037	
	r							
	LD.	-.029395	.0105773	-2.78	0.005	-.0501261	-.0086639	
	lm							
	LD.	.4472679	.0956691	4.68	0.000	.2597599	.634776	
	_cons	.0071036	.0016835	4.22	0.000	.0038039	.0104033	
-----								

. vargranger

Granger Cuasality Test (グレンジャー因果性テスト)

Granger causality Wald tests

Equation	Excluded	chi2	df	Prob > chi2
D_lgdp	D_r	.14297	1	0.705
D_lgdp	D_lm	.0196	1	0.889
D_lgdp	ALL	.15705	2	0.924
D_r	D_lgdp	.22731	1	0.634
D_r	D_lm	.04356	1	0.835
D_r	ALL	.3039	2	0.859
D_lm	D_lgdp	4.0064	1	0.045
D_lm	D_r	7.7232	1	0.005
D_lm	ALL	10.798	2	0.005

gdp から m への因果関係 (p 値 0.045)

r から m への因果関係 (p 値 0.005)

学部の教科書レベルの貨幣需要関数

$m=f(\text{gdp}, r)$ , すなわち, m は gdp と r の関数