

### 7.3 Impulse Response Function (インパルス応答関数):

$$\frac{\partial y_{i,t+m}}{\partial \epsilon_{j,t}}, \quad m = 1, 2, \dots,$$

where  $i, j = 1, 2, \dots, k$ .

#### **Example: AR( $p$ ) Process:**

When  $y_t$  is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)^{-1} \epsilon_t \\ &= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots \end{aligned}$$

The impulse response function is:

$$\frac{\partial y_{i,t+k}}{\partial \epsilon_{j,t}} = \theta_{ij,k}, \quad k = 1, 2, \dots,$$

where  $\theta_{i,j,k}$  denotes the  $(i, j)$ th element of  $\theta_k$ .

$$\begin{aligned}y_t &= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots \\ &= PP^{-1} \epsilon_t + \theta_1 PP^{-1} \epsilon_{t-1} + \theta_2 PP^{-1} \epsilon_{t-2} + \cdots \\ &= \Omega_0 \eta_t + \Omega_1 \eta_{t-1} + \Omega_2 \eta_{t-2} + \cdots,\end{aligned}$$

where  $V(\eta_t) = I_k$ , and  $\Omega_i = \theta_i P$  for  $i = 0, 1, 2, \dots$  and  $\Omega_0 = P$ .

$$\frac{\partial y_{i,t+m}}{\partial \eta_{j,t}}, \quad m = 1, 2, \dots,$$

where  $i, j = 1, 2, \dots, k$ .

⇒ **Orthogonalized Impulse Response Function** (直交化インパルス応答関数)

**Example:**

```
. varbasic d.lgdp d.r d.lm, lags(1)
```

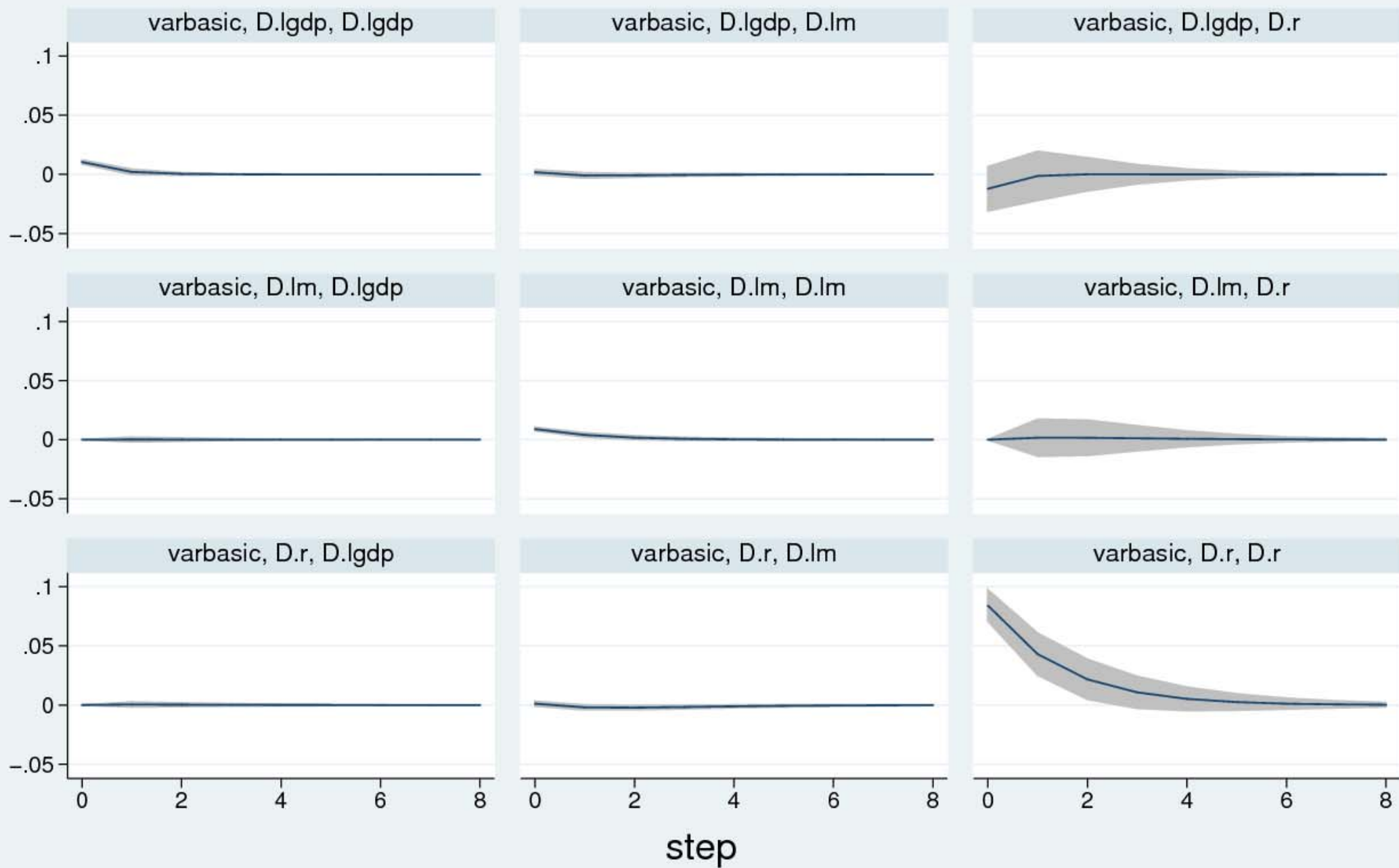
Vector autoregression

|                |            |            |   |           |
|----------------|------------|------------|---|-----------|
| Sample:        | 3 - 81     | No. of obs | = | 79        |
| Log likelihood | = 592.2334 | AIC        | = | -14.68945 |
| FPE            | = 8.38e-11 | HQIC       | = | -14.54526 |
| Det(Sigma_ml)  | = 6.18e-11 | SBIC       | = | -14.32954 |

| Equation | Parms | RMSE    | R-sq   | chi2     | P>chi2 |
|----------|-------|---------|--------|----------|--------|
| D_lgdp   | 4     | .010717 | 0.0422 | 3.480972 | 0.3232 |
| D_r      | 4     | .087186 | 0.2553 | 27.0782  | 0.0000 |
| D_lm     | 4     | .009434 | 0.2903 | 32.30929 | 0.0000 |

|        | Coef.    | Std. Err. | z    | P> z  | [95% Conf. Interval] |          |
|--------|----------|-----------|------|-------|----------------------|----------|
| D_lgdp |          |           |      |       |                      |          |
| lgdp   |          |           |      |       |                      |          |
| LD.    | .2031129 | .1119361  | 1.81 | 0.070 | -.0162778            | .4225037 |
| r      |          |           |      |       |                      |          |
| LD.    | .0045431 | .0120151  | 0.38 | 0.705 | -.0190061            | .0280922 |
| lm     |          |           |      |       |                      |          |
| LD.    | .0152162 | .1086739  | 0.14 | 0.889 | -.1977807            | .228213  |
| _cons  | .0019504 | .0019124  | 1.02 | 0.308 | -.0017978            | .0056986 |
| D_r    |          |           |      |       |                      |          |
| lgdp   |          |           |      |       |                      |          |
| LD.    | .4341641 | .9106374  | 0.48 | 0.634 | -1.350652            | 2.218981 |
| r      |          |           |      |       |                      |          |

|       |       |           |          |       |       |           |           |
|-------|-------|-----------|----------|-------|-------|-----------|-----------|
|       | LD.   | .5085677  | .0977469 | 5.20  | 0.000 | .3169874  | .7001481  |
|       | lm    |           |          |       |       |           |           |
|       | LD.   | .1845222  | .8840978 | 0.21  | 0.835 | -1.548278 | 1.917322  |
|       | _cons | -.0202984 | .0155578 | -1.30 | 0.192 | -.0507912 | .0101943  |
| ----- |       |           |          |       |       |           |           |
| D_lm  |       |           |          |       |       |           |           |
|       | lgdp  |           |          |       |       |           |           |
|       | LD.   | -.1972406 | .098541  | -2.00 | 0.045 | -.3903774 | -.0041037 |
|       | r     |           |          |       |       |           |           |
|       | LD.   | -.029395  | .0105773 | -2.78 | 0.005 | -.0501261 | -.0086639 |
|       | lm    |           |          |       |       |           |           |
|       | LD.   | .4472679  | .0956691 | 4.68  | 0.000 | .2597599  | .634776   |
|       | _cons | .0071036  | .0016835 | 4.22  | 0.000 | .0038039  | .0104033  |
| ----- |       |           |          |       |       |           |           |



Graphs by irfname, impulse variable, and response variable

## 8 Unit Root (単位根) and Cointegration (共和分)

### 8.1 Unit Root (単位根) Test (Dickey-Fuller (DF) Test)

1. Why is a unit root problem important?

(a) Economic variables increase over time in general.

One of the assumptions of OLS is stationarity on  $y_t$  and  $x_t$ .

This assumption implies that  $\frac{1}{T}X'X$  converges to a fixed matrix as  $T$  is large.

That is, asymptotic normality of OLS estimator goes not hold.

(b) In nonstationary time series, the unit root is the most important.

In the case of unit root, OLSE of the first-order autoregressive coefficient is consistent.

OLSE is  $\sqrt{T}$ -consistent in the case of stationary AR(1) process, but OLSE is  $T$ -consistent in the case of nonstationary AR(1) process.

(c) A lot of economic variables increase over time.

It is important to check an economic variable is trend stationary (i.e.,  $y_t = a_0 + a_1t + \epsilon_t$ ) or difference stationary (i.e.,  $y_t = b_0 + y_{t-1} + \epsilon_t$ ).

Consider  $k$ -step ahead prediction for both cases.

$$\text{(Trend Stationarity)} \quad y_{t+k|t} = a_0 + a_1(t+k)$$

$$\text{(Difference Stationarity)} \quad y_{t+k|t} = b_0k + y_t$$

## 2. The Case of $|\phi_1| < 1$ :

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \quad y_0 = 0, \quad t = 1, \dots, T$$

Then, OLSE of  $\phi_1$  is:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^T y_{t-1}y_t}{\sum_{t=1}^T y_{t-1}^2}.$$

In the case of  $|\phi_1| < 1$ ,

$$\hat{\phi}_1 = \phi_1 + \frac{\frac{1}{T} \sum_{t=1}^T y_{t-1}\epsilon_t}{\frac{1}{T} \sum_{t=1}^T y_{t-1}^2} \longrightarrow \phi_1 + \frac{E(y_{t-1}\epsilon_t)}{E(y_{t-1}^2)} = \phi_1.$$

Note as follows:

$$\frac{1}{T} \sum_{t=1}^T y_{t-1}\epsilon_t \longrightarrow E(y_{t-1}\epsilon_t) = 0.$$



By the central limit theorem,

$$\frac{\bar{y}_\epsilon - \mathbb{E}(\bar{y}_\epsilon)}{\sqrt{\mathbb{V}(\bar{y}_\epsilon)}} \longrightarrow N(0, 1)$$

where

$$\bar{y}_\epsilon = \frac{1}{T} \sum_{t=1}^T y_{t-1} \epsilon_t.$$

$$\mathbb{E}(\bar{y}_\epsilon) = 0,$$

$$\begin{aligned} \mathbb{V}(\bar{y}_\epsilon) &= \mathbb{V}\left(\frac{1}{T} \sum_{t=1}^T y_{t-1} \epsilon_t\right) = \mathbb{E}\left(\left(\frac{1}{T} \sum_{t=1}^T y_{t-1} \epsilon_t\right)^2\right) \\ &= \frac{1}{T^2} \mathbb{E}\left(\sum_{t=1}^T \sum_{s=1}^T y_{t-1} y_{s-1} \epsilon_t \epsilon_s\right) = \frac{1}{T^2} \mathbb{E}\left(\sum_{t=1}^T y_{t-1}^2 \epsilon_t^2\right) = \frac{1}{T} \sigma^2 \gamma(0). \end{aligned}$$

Therefore,

$$\frac{\bar{y}_\epsilon}{\sqrt{\sigma^2 \gamma(0)/T}} = \frac{1}{\sigma_\epsilon \sqrt{\gamma(0)}} \frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t \longrightarrow N(0, 1),$$

which is rewritten as:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t \rightarrow N(0, \sigma_\epsilon^2 \gamma(0)).$$

Using  $\frac{1}{T} \sum_{t=1}^T y_{t-1}^2 \rightarrow E(y_{t-1}^2) = \gamma(0)$ , we have the following asymptotic distribution:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t}{\frac{1}{T} \sum_{t=1}^T y_{t-1}^2} \rightarrow N\left(0, \frac{\sigma_\epsilon^2}{\gamma(0)}\right) = N\left(0, 1 - \phi_1^2\right).$$

Note that  $\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$ .

3. In the case of  $\phi_1 = 1$ , as expected, we have:

$$\sqrt{T}(\hat{\phi}_1 - 1) \rightarrow 0.$$

That is,  $\hat{\phi}_1$  has the distribution which converges in probability to  $\phi_1 = 1$  (i.e., degenerated distribution).

Is this true?

4. **The Case of  $\phi_1 = 1$ :**  $\implies$  Random Walk Process

$y_t = y_{t-1} + \epsilon_t$  with  $y_0 = 0$  is written as:

$$y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots + \epsilon_1.$$

Therefore, we can obtain:

$$y_t \sim N(0, \sigma_\epsilon^2 t).$$

The variance of  $y_t$  depends on time  $t$ .  $\implies y_t$  is nonstationary.

5. Remember that  $\hat{\phi}_1 = \phi_1 + \frac{\sum y_{t-1} \epsilon_t}{\sum y_{t-1}^2}$ .

(a) First, consider the numerator  $\sum y_{t-1}\epsilon_t$ .

$$\text{We have } y_t^2 = (y_{t-1} + \epsilon_t)^2 = y_{t-1}^2 + 2y_{t-1}\epsilon_t + \epsilon_t^2.$$

Therefore, we obtain:

$$y_{t-1}\epsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \epsilon_t^2).$$

Taking into account  $y_0 = 0$ , we have:

$$\sum_{t=1}^T y_{t-1}\epsilon_t = \frac{1}{2}y_T^2 - \frac{1}{2}\sum_{t=1}^T \epsilon_t^2.$$

Divided by  $\sigma_\epsilon^2 T$  on both sides, we have the following:

$$\frac{1}{\sigma_\epsilon^2 T} \sum_{t=1}^T y_{t-1}\epsilon_t = \frac{1}{2} \left( \frac{y_T}{\sigma_\epsilon \sqrt{T}} \right)^2 - \frac{1}{2\sigma_\epsilon^2 T} \sum_{t=1}^T \epsilon_t^2.$$

From  $y_t \sim N(0, \sigma_\epsilon^2 t)$ , we obtain the following result:

$$\left( \frac{y_T}{\sigma_\epsilon \sqrt{T}} \right)^2 \sim \chi^2(1).$$

Moreover, the second term is derived from:

$$\frac{1}{T} \sum_{t=1}^T \epsilon_t^2 \rightarrow \sigma_\epsilon^2.$$

Therefore,

$$\frac{1}{\sigma_\epsilon^2 T} \sum_{t=1}^T y_{t-1} \epsilon_t = \frac{1}{2} \left( \frac{y_T}{\sigma \sqrt{T}} \right)^2 - \frac{1}{2\sigma_\epsilon^2} \frac{1}{T} \sum_{t=1}^T \epsilon_t^2 \rightarrow \frac{1}{2} (\chi^2(1) - 1).$$

(b) Next, consider  $\sum y_{t-1}^2$ .

$$\mathbb{E} \left( \sum_{t=1}^T y_{t-1}^2 \right) = \sum_{t=1}^T \mathbb{E}(y_{t-1}^2) = \sum_{t=1}^T \sigma_\epsilon^2 (t-1) = \sigma_\epsilon^2 \frac{T(T-1)}{2}.$$

Thus, we obtain the following result:

$$\frac{1}{T^2} \mathbb{E} \left( \sum_{t=1}^T y_{t-1}^2 \right) \rightarrow \text{a fixed value.}$$

Therefore,

$$\frac{1}{T^2} \sum_{t=1}^T y_{t-1}^2 \longrightarrow \text{a distribution.}$$

6. Summarizing the results up to now,  $T(\hat{\phi}_1 - \phi_1)$ , not  $\sqrt{T}(\hat{\phi}_1 - \phi_1)$ , has limiting distribution in the case of  $\phi_1 = 1$ .

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T) \sum y_{t-1} \epsilon_t}{(1/T^2) \sum y_{t-1}^2} \longrightarrow \text{a distribution.}$$

The distributions of the  $t$  statistic:  $\frac{\hat{\phi}_1 - 1}{s_\phi}$ , where  $s_\phi$  denotes the standard error of  $\hat{\phi}_1$ .

⇒ Compare  $t$  distribution with (a) – (c).

⇒ **Unit Root Test (単位根検定, or Dickey-Fuller (DF) Test)**

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

Test  $H_0 : \phi_1 = 1$  against  $H_1 : \phi_1 < 1$ .

Equivalently,

$$\Delta y_t = \rho y_{t-1} + \epsilon_t.$$

Test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ .

***t* Distribution**

| <i>T</i> | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 25       | -2.49 | -2.06 | -1.71 | -1.32 | 1.32  | 1.71  | 2.06  | 2.49  |
| 50       | -2.40 | -2.01 | -1.68 | -1.30 | 1.30  | 1.68  | 2.01  | 2.40  |
| 100      | -2.36 | -1.98 | -1.66 | -1.29 | 1.29  | 1.66  | 1.98  | 2.36  |
| 250      | -2.34 | -1.97 | -1.65 | -1.28 | 1.28  | 1.65  | 1.97  | 2.34  |
| 500      | -2.33 | -1.96 | -1.65 | -1.28 | 1.28  | 1.65  | 1.96  | 2.33  |
| $\infty$ | -2.33 | -1.96 | -1.64 | -1.28 | 1.28  | 1.64  | 1.96  | 2.33  |



$$(a) H_0 : y_t = y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

| $T$      | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 25       | -2.66 | -2.26 | -1.95 | -1.60 | 0.92  | 1.33  | 1.70  | 2.16  |
| 50       | -2.62 | -2.25 | -1.95 | -1.61 | 0.91  | 1.31  | 1.66  | 2.08  |
| 100      | -2.60 | -2.24 | -1.95 | -1.61 | 0.90  | 1.29  | 1.64  | 2.03  |
| 250      | -2.58 | -2.23 | -1.95 | -1.62 | 0.89  | 1.29  | 1.63  | 2.01  |
| 500      | -2.58 | -2.23 | -1.95 | -1.62 | 0.89  | 1.28  | 1.62  | 2.00  |
| $\infty$ | -2.58 | -2.23 | -1.95 | -1.62 | 0.89  | 1.28  | 1.62  | 2.00  |

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.

$$(b) H_0 : y_t = y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

| $T$      | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 25       | -3.75 | -3.33 | -3.00 | -2.63 | -0.37 | 0.00  | 0.34  | 0.72  |
| 50       | -3.58 | -3.22 | -2.93 | -2.60 | -0.40 | -0.03 | 0.29  | 0.66  |
| 100      | -3.51 | -3.17 | -2.89 | -2.58 | -0.42 | -0.05 | 0.26  | 0.63  |
| 250      | -3.46 | -3.14 | -2.88 | -2.57 | -0.42 | -0.06 | 0.24  | 0.62  |
| 500      | -3.44 | -3.13 | -2.87 | -2.57 | -0.43 | -0.07 | 0.24  | 0.61  |
| $\infty$ | -3.43 | -3.12 | -2.86 | -2.57 | -0.44 | -0.07 | 0.23  | 0.60  |

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \alpha_0 + \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.

$$(c) H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1$$

| $T$      | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 25       | -4.38 | -3.95 | -3.60 | -3.24 | -1.14 | -0.80 | -0.50 | -0.15 |
| 50       | -4.15 | -3.80 | -3.50 | -3.18 | -1.19 | -0.87 | -0.58 | -0.24 |
| 100      | -4.04 | -3.73 | -3.45 | -3.15 | -1.22 | -0.90 | -0.62 | -0.28 |
| 250      | -3.99 | -3.69 | -3.43 | -3.13 | -1.23 | -0.92 | -0.64 | -0.31 |
| 500      | -3.98 | -3.68 | -3.42 | -3.13 | -1.24 | -0.93 | -0.65 | -0.32 |
| $\infty$ | -3.96 | -3.66 | -3.41 | -3.12 | -1.25 | -0.94 | -0.66 | -0.33 |

To test  $H_0 : \rho = 0$  against  $H_1 : \rho < 0$ , estimate  $\Delta y_t = \alpha_0 + \alpha_1 t + \rho y_{t-1} + \epsilon_t$  and compare the  $t$ -value of  $\rho$  with the above table.