## 8.2 Unit Root (More Formally)

Consider $y_t = y_{t-1} + \epsilon_t$ and $y_0 = 0$.

$$y_t = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_t \sim N(0, t\sigma^2)$$

$$\frac{1}{\sqrt{T}} y_t = \frac{1}{\sqrt{T}}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t) \sim N(0, \frac{t}{T}\sigma^2) \longrightarrow N(0, r\sigma^2)$$

where $0 \leq r \leq 1$ and $r = \dfrac{t}{T}$.

Note that time interval $(1,\ T)$ is transformed into $(0,\ 1)$, divided by $T$.

$$\frac{1}{\sqrt{T}\sigma} y_t = \frac{1}{\sqrt{T}\sigma}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t) \sim N(0, \frac{t}{T}) \longrightarrow N(0, r) \equiv W(r)$$

As $T$ ($t$ at the same time) goes to infinity keeping $r = \dfrac{t}{T}$, $W(r)$ results in a continuous function of $r$ where $r$ takes any number between zero and one.

$W(r)$ is a normal random variable with mean zero and variance $r$ and it is called the **Brownian motion**.

Moreover, we consider the following:

$$\frac{1}{\sqrt{T}\sigma} y_{t'} = \frac{1}{\sqrt{T}\sigma}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{t'}) \longrightarrow N(0, r') \equiv W(r').$$

For $t' > t$, we have the following:

$$\begin{aligned}\frac{1}{\sqrt{T}\sigma} y_{t'} &= \frac{1}{\sqrt{T}\sigma}(\epsilon_1 + \cdots + \epsilon_t + \epsilon_{t+1} + \cdots \epsilon_{t'}) \longrightarrow N(0, r') \equiv W(r') \\ &= \frac{1}{\sqrt{T}\sigma} y_t + \frac{1}{\sqrt{T}\sigma}(\epsilon_{t+1} + \cdots \epsilon_{t'}).\end{aligned}$$

Therefore, we have

$$\frac{1}{\sqrt{T}\sigma} y_{t'} - \frac{1}{\sqrt{T}\sigma} y_t = \frac{1}{\sqrt{T}\sigma}(\epsilon_{t+1} + \cdots \epsilon_{t'}) \longrightarrow N(0, r' - r) \equiv W(r') - W(r).$$

That is, $W(r)$ is independent of $W(r') - W(r)$ for $r' > r$.

Moreover, note as follows:

$$\frac{1}{T\sqrt{T}\sigma}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T}\Big(\frac{y_t}{\sqrt{T}\sigma}\Big) \longrightarrow \int_0^1 W(r)\mathrm{d}r$$

where $\dfrac{1}{T}$ and $\displaystyle\sum_{t=1}^{T}$ are replaced by d$r$ and $\displaystyle\int_0^1$ as $T$ goes to infinity.

We devide the time interval $(0,\ 1)$ into $T$ time intervals $\Big(\dfrac{t}{T},\ \dfrac{t+1}{T}\Big)$.

That is, time interval $(1,\ T)$ is transformed into (0, 1).

(*) We know that $\dfrac{y_t}{\sqrt{T}\sigma} \longrightarrow W(r)$ as $\dfrac{t}{T} \longrightarrow r$.

**Summary:** **Properties of $W(r)$ for $0 < r < 1$:**

1. $W(r) \equiv N(0, r) \quad \Longrightarrow \quad W(r)$ is a random variable.

2. $W(1) \equiv N(0, 1)$

3. $W(1)^2 \equiv \chi^2(1) \quad \Longrightarrow$ Remember that $Z^2 \sim \chi^2(1)$ when $Z \sim N(0, 1)$.

4. $W(r)$ is independent of $W(r') - W(r)$ for $r < r'$.

5. $W(r_4) - W(r_3)$ is independent of $W(r_2) - W(r_1)$ for $0 \le r_1 < r_2 < r_3 < r_4 \le 1$.

   $\Longrightarrow$ The interval between $r_4$ and $r_3$ is not overlapped with the interval between $r_2$ and $r_1$.

- **True Model $y_t = y_{t-1} + \epsilon_t$ vs Estimated Model $y_t = \phi y_{t-1} + \epsilon_t$:** Under $\phi = 1$,

we estimate $\phi$ in the regression model:

$$y_t = \phi y_{t-1} + \epsilon_t$$

OLS of $\phi$ is:

$$\hat{\phi} = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2} = \phi + \frac{\sum_{t=1}^T y_{t-1}\epsilon_t}{\sum_{t=1}^T y_{t-1}^2}$$

As mentioned aove, the numerator is related to:

$$\frac{1}{\sigma^2 T}\sum_{t=1}^T y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma\sqrt{T}}\right)^2 - \frac{1}{2\sigma^2}\frac{1}{T}\sum_{t=1}^T \epsilon_t^2 \longrightarrow \frac{1}{2}W(1)^2 - \frac{1}{2}$$

which is rewritten by using the Brownian motion $W(1)$.

The denominator is:

$$\frac{1}{\sigma^2 T^2}\sum_{t=1}^T y_{t-1}^2 \approx \frac{1}{\sigma^2 T^2}\sum_{t=1}^T y_t^2 = \frac{1}{T}\sum_{t=1}^T \left(\frac{y_t}{\sigma\sqrt{T}}\right)^2 \longrightarrow \int_0^1 W(r)^2 \mathrm{d}r$$

where $\frac{1}{T} \longrightarrow \mathrm{d}r$ and $\frac{y_t}{\sigma\sqrt{T}} \longrightarrow W(r)$ for $\frac{t}{T} \longrightarrow r$.

Thus, under $\phi = 1$, $T(\hat{\phi} - \phi)$ is asymptotically distributed as follows:

$$T(\hat{\phi} - \phi) = T(\hat{\phi} - 1) = \frac{\frac{1}{\sigma^2 T} \sum_{t=1}^{T} y_{t-1}\epsilon_t}{\frac{1}{\sigma^2 T^2} \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}\Big(W(1)^2 - 1\Big)}{\int_0^1 W(r)^2 \mathrm{d}r}$$

- $t$ value:

In the regression model: $y_t - y_{t-1} \equiv \Delta y_t = \rho y_{t-1} + \epsilon_t$, OLSE of $\rho = \phi - 1$ is given by $\hat{\rho} = \hat{\phi} - 1$.

$t$ value is $\frac{\hat{\rho}}{s_\rho} = \frac{\hat{\phi} - 1}{s_\phi}$, where $s_\rho$ and $s_\phi$ denote the standard errors of $\hat{\rho}$ and $\hat{\phi}$.
Note that $s_\rho = s_\phi$ because of $\mathrm{V}(\hat{\rho}) = \mathrm{V}(\hat{\phi} - 1) = \mathrm{V}(\hat{\phi})$.

The standard error of $\hat{\phi}$, denoted by $s_\phi$, is given by: $s_\phi^2 = \dfrac{s^2}{\sum_{t=1}^T y_{t-1}^2}$, where $s^2 = \dfrac{1}{T}\sum_{t=1}^T (y_t - \hat{\phi} y_{t-1})^2$, called the standard error of regression.

$$
\begin{aligned}
s^2 &= \frac{1}{T}\sum_{t=1}^T (y_t - \hat{\phi} y_{t-1})^2 \\
&= \frac{1}{T}\sum_{t=1}^T (\epsilon_t - (\hat{\phi} - 1) y_{t-1})^2 \\
&= \frac{1}{T}\sum_{t=1}^T \epsilon_t^2 - 2\frac{1}{T}\frac{1}{T}(\hat{\phi} - 1)\sum_{t=1}^T y_{t-1}\epsilon_t + \frac{1}{T}(\hat{\phi} - 1)^2 \sum_{t=1}^T y_{t-1}^2 \\
&= \frac{1}{T}\sum_{t=1}^T \epsilon_t^2 - 2\frac{\sigma^2}{T}\Big[T(\hat{\phi} - 1)\Big]\Big[\frac{1}{T\sigma^2}\sum_{t=1}^T y_{t-1}\epsilon_t\Big] + \frac{\sigma^2}{T}\Big[T(\hat{\phi} - 1)\Big]^2\Big[\frac{1}{T^2\sigma^2}\sum_{t=1}^T y_{t-1}^2\Big] \\
&\longrightarrow \sigma^2.
\end{aligned}
$$

The random variables in $\big[\cdot\big]$ converge in distribution..

Note that in the right hand side of the fourth line the second and third terms go to zero, because we know the followings:

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 &\longrightarrow \sigma^2 \\
T(\hat{\phi}-1) &\longrightarrow \frac{\frac{1}{2}\Big(W(1)^2-1\Big)}{\int_0^1 W(r)^2 \mathrm{d}r} \\
\frac{1}{T\sigma^2}\sum_{t=1}^{T} y_{t-1}\epsilon_t &\longrightarrow \frac{1}{2}W(1)^2-\frac{1}{2} \\
\frac{1}{T^2\sigma^2}\sum_{t=1}^{T} y_t^2 &\longrightarrow \int_0^1 W(r)^2 \mathrm{d}r
\end{aligned}$$

Therefore, from $s_\phi^2 = \dfrac{1}{T^2\sigma^2}\dfrac{s^2}{\frac{1}{T^2\sigma^2}\sum_{t=1}^{T} y_{t-1}^2}$, we obtain $T^2 s_\phi^2 \longrightarrow \Big(\int_0^1 W(r)^2 \mathrm{d}r\Big)^{-1}$.

$t$ value is given by:

$$\frac{\hat{\phi}-1}{s_\phi} = \frac{T(\hat{\phi}-1)}{Ts_\phi} \longrightarrow \frac{\frac{1}{2}\Big(W(1)^2-1\Big)/\int_0^1 W(r)^2 \mathrm{d}r}{\Big(\int_0^1 W(r)^2 \mathrm{d}r\Big)^{-1/2}} = \frac{\frac{1}{2}\Big(W(1)^2-1\Big)}{\Big(\int_0^1 W(r)^2 \mathrm{d}r\Big)^{1/2}},$$

which is not a normal distribution.

- **True Model $y_t = y_{t-1} + \epsilon_t$ vs Estimated Model $y_t = \alpha + \phi y_{t-1} + \epsilon_t$:** Under $\alpha = 0$ and $\phi = 1$, we estimate $\alpha$ and $\phi$ in the regression model:

$$y_t = \alpha + \phi y_{t-1} + \epsilon_t$$

OLSEs of $\alpha$ and $\phi$ are:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_t \\ \sum y_{t-1} y_t \end{pmatrix} = \begin{pmatrix} \alpha \\ \phi \end{pmatrix} + \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1} \epsilon_t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ \phi \end{pmatrix} + \frac{1}{T \sum y_{t-1}^2 - (\sum y_{t-1})^2} \begin{pmatrix} \sum y_{t-1}^2 & -\sum y_{t-1} \\ -\sum y_{t-1} & T \end{pmatrix} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1} \epsilon_t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ \phi \end{pmatrix} + \frac{1}{T \sum y_{t-1}^2 - (\sum y_{t-1})^2} \begin{pmatrix} (\sum y_{t-1}^2)(\sum \epsilon_t) - (\sum y_{t-1})(\sum y_{t-1} \epsilon_t) \\ -(\sum y_{t-1})(\sum \epsilon_t) + T(\sum y_{t-1} \epsilon_t) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ \phi \end{pmatrix} + \begin{pmatrix} \dfrac{(\sum y_t^2)(\frac{1}{T}\sum \epsilon_t) - \overline{y}\sum y_{t-1}\epsilon_t}{\sum y_t^2 - T\overline{y}^2} \\ \dfrac{-\overline{y}\sum \epsilon_t + \sum y_{t-1}\epsilon_t}{\sum y_t^2 - T\overline{y}^2} \end{pmatrix}$$

Note that $\frac{1}{T}\sum y_{t=1} \approx \frac{1}{T}\sum y_t = \overline{y}$ and $\sum y_{t-1}^2 \approx \sum y_t^2$ for large $T$.

In the true model, $\alpha = 0$ and $\phi = 1$.

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\phi} - 1 \end{pmatrix} = \begin{pmatrix} \dfrac{(\sum y_t^2)(\frac{1}{T}\sum \epsilon_t) - \overline{y}\sum y_{t-1}\epsilon_t}{\sum y_t^2 - T\overline{y}^2} \\ \dfrac{-\overline{y}\sum \epsilon_t + \sum y_{t-1}\epsilon_t}{\sum y_t^2 - T\overline{y}^2} \end{pmatrix}$$

For each element of the vector, we consider each term in the numerator and denominator.

- $\sum_{t=1}^{T} y_{t-1}\epsilon_t$:

Taking the square of $y_t = y_{t-1} + \epsilon_t$ on both sides, we obtain: $y_t^2 = y_{t-1}^2 + y_{t-1}\epsilon_t + \epsilon_t^2$.

Then, we can rewrite as: $y_{t-1}\epsilon_t = \dfrac{1}{2}(y_t^2 - y_{t-1}^2 - \epsilon_t^2)$ for $y_0 = 0$.

Taking a sum from $t = 1$ to $T$, we have:

$$\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\sum_{t=1}^{T}(y_t^2 - y_{t-1}^2 - \epsilon_t^2) = \frac{1}{2}y_T^2 - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t^2,$$

which is divided by $T\sigma^2$ on both sides, then we obtain:

$$\frac{1}{T\sigma^2}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\Big(\frac{y_T}{\sqrt{T}\sigma}\Big)^2 - \frac{1}{2\sigma^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \frac{1}{2}W(1)^2 - \frac{1}{2}.$$

Note that $\dfrac{y_T}{\sqrt{T}\sigma} = W(1)$ and $\dfrac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \mathrm{E}(\epsilon_t^2) = \sigma^2$.

- $\overline{y}$:

Note that $\dfrac{1}{T}\displaystyle\sum_{t=1}^{T} y_t = \overline{y}$ and $\dfrac{1}{T}\displaystyle\sum_{t=1}^{T} y_{t-1} = \overline{y}$. We can rewrite $\overline{y}$ as follows:

$$\overline{y} = \frac{1}{T}\sum_{t=1}^{T} y_t = \sqrt{T}\sigma\frac{1}{T}\sum_{t=1}^{T}\frac{y_t}{\sqrt{T}\sigma}$$

which is rewritten as:

$$\frac{\overline{y}}{\sqrt{T}\sigma} = \frac{1}{T}\sum_{t=1}^{T}\frac{y_t}{\sqrt{T}\sigma} \longrightarrow \int_0^1 W(r)\mathrm{d}r$$

Note that $\dfrac{y_t}{\sqrt{T}\sigma} \longrightarrow W(r)$ as $\dfrac{t}{T} \longrightarrow r$.

- $\sum_{t=1}^{T} \epsilon_t$:

From $y_T = \sum_{t=1}^{T} \epsilon_t$, we have: $\dfrac{1}{\sqrt{T}\sigma} \sum_{t=1}^{T} \epsilon_t = \dfrac{y_T}{\sqrt{T}\sigma} = W(1)$.

- $\sum_{t=1}^{T} y_{t-1}^2$:

From $\sum_{t=1}^{T} y_{t-1}^2 \approx \sum_{t=1}^{T} y_t^2$, we obtain: $\dfrac{1}{T} \sum_{t=1}^{T} \Big(\dfrac{y_t}{\sqrt{T}\sigma}\Big)^2 \longrightarrow \int_0^1 W(r)^2 \mathrm{d}r$.

Note that $\dfrac{y_t}{\sqrt{T}\sigma} \longrightarrow W(r)$ as $\dfrac{t}{T} \longrightarrow r$.

Thus, $\hat{\phi} - 1 = \dfrac{\sum y_{t-1}\epsilon_t - \bar{y}\sum \epsilon_t}{\sum y_t^2 - T\bar{y}^2}$ is rerwitten as:

$$T(\hat{\phi} - 1) = \frac{\frac{1}{T\sigma^2}(\sum y_{t-1}\epsilon_t - \bar{y}\sum \epsilon_t)}{\frac{1}{T^2\sigma^2}(\sum y_t^2 - T\bar{y}^2)} = \frac{\frac{1}{T\sigma^2}\sum y_{t-1}\epsilon_t - \left(\frac{\bar{y}}{\sqrt{T}\sigma}\right)\left(\frac{1}{\sqrt{T}\sigma}\sum \epsilon_t\right)}{\frac{1}{T}\sum\left(\frac{y_t}{\sqrt{T}\sigma}\right)^2 - \left(\frac{\bar{y}}{\sqrt{T}\sigma}\right)^2}$$

$$\longrightarrow \frac{\frac{1}{2}\left(W(1)^2 - 1\right) - W(1)\int_0^1 W(r)\mathrm{d}r}{\int_0^1 W(r)^2\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r\right)^2}$$

Remember that OLSE of $\alpha$ is given by:

$$\hat{\alpha} = \alpha + \frac{(\sum y_t^2)(\frac{1}{T}\sum \epsilon_t) - \bar{y}\sum y_{t-1}\epsilon_t}{\sum y_t^2 - T\bar{y}^2}$$

Under $\alpha = 0$, $\hat{\alpha}$ is rewritten as follows:

$$\sqrt{T}\hat{\alpha} = \frac{\sigma\Big(\frac{1}{T}\sum\big(\frac{y_t}{\sqrt{T}\sigma}\big)^2\Big)\Big(\frac{1}{\sqrt{T}\sigma}\sum\epsilon_t\Big) - \sigma\Big(\frac{\bar{y}}{\sqrt{T}\sigma}\Big)\Big(\frac{1}{T\sigma^2}\sum y_{t-1}\epsilon_t\Big)}{\frac{1}{T}\sum\big(\frac{y_t}{\sqrt{T}\sigma}\big)^2 - \big(\frac{\bar{y}}{\sqrt{T}\sigma}\big)^2}$$

$$\longrightarrow \frac{\sigma W(1)\int_0^1 W(r)^2 \mathrm{d}r - \sigma\frac{1}{2}(W(1)^2 - 1)\int_0^1 W(r)\mathrm{d}r}{\int_0^1 W(r)^2\mathrm{d}r - \Big(\int_0^1 W(r)\mathrm{d}r\Big)^2}$$

Thus, convergence speed of $\hat{\phi}$ is different from that of $\hat{\alpha}$.

Neither $\sqrt{T}\hat{\alpha}$ nor $T(\hat{\phi} - 1)$ are normal.