8.3 Serially Correlated Errors

Consider the case where the error term is serially correlated.

8.3.1 Augmented Dickey-Fuller (ADF) Test

Consider the following AR(p) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \qquad \epsilon_t \sim \text{iid}(0, \sigma^2),$$

which is rewritten as: $\qquad \phi(L) y_t = \epsilon_t.$

When the above model has a unit root, we have $\phi(1) = 0$, i.e., $\phi_1 + \phi_2 + \cdots + \phi_p = 1$.

The above AR(p) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + +\delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$ and $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$.

The null and alternative hypotheses are:

$$H_0: \ \rho = 1 \text{ (Unit root)},$$

$$H_1: \ \rho < 1 \text{ (Stationary)}.$$

Use the t test, where we have the same asymptotic distributions.

We can utilize the same tables as before.

Choose p by AIC or SBIC.

Use $N(0, 1)$ to test $H_0: \ \delta_j = 0$ against $H_1: \ \delta_j \neq 0$ for $j = 1, 2, \cdots, p-1$.

Reference

Kurozumi (2008) "Economic Time Series Analysis and Unit Root Tests: Development and Perspective," *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.

Download the above paper from:

http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html

Example of ADF Test

```
. gen time=_n

. tsset time
        time variable:  time, 1 to 516
                delta:  1 unit

. gen sexpend=expend-l12.expend
(12 missing values generated)

. corrgram sexpend

                                                  -1       0       1 -1       0       1
 LAG       AC       PAC      Q     Prob>Q  [Autocorrelation]  [Partial Autocor]
-------------------------------------------------------------------------------
1        0.7177   0.7184   261.14  0.0000          |-----            |-----
2        0.7036   0.3895    512.6  0.0000          |-----            |---
3        0.7031   0.2817   764.23  0.0000          |-----            |--
4        0.6366   0.0456   970.94  0.0000          |-----            |
5        0.6413   0.1116   1181.1  0.0000          |-----            |
6        0.6267   0.0815   1382.2  0.0000          |-----            |
7        0.6208   0.0972     1580  0.0000          |----             |
8        0.6384   0.1286   1789.5  0.0000          |-----            |-
9        0.5926  -0.0205   1970.5  0.0000          |----             |
10       0.5847  -0.0014   2146.9  0.0000          |----             |
11       0.5658  -0.0185   2312.6  0.0000          |----             |
12       0.4529  -0.2570   2418.9  0.0000          |---            --|
13       0.5601   0.2318   2581.8  0.0000          |----             |-
14       0.5393   0.1095   2733.2  0.0000          |----             |
15       0.5277   0.0850   2878.4  0.0000          |----             |
```

```
. varsoc d.sexpend, exo(l.sexpend) maxlag(25)
```

Selection-order criteria
Sample: 39 - 516 Number of obs = 478

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	-4917.7				5.1e+07	20.5845	20.5914	20.6019
1	-4878.69	78.013	1	0.000	4.3e+07	20.4255	20.4358	20.4516
2	-4858.95	39.481	1	0.000	4.0e+07	20.3471	20.3608	20.382
3	-4858.46	.97673	1	0.323	4.0e+07	20.3492	20.3664	20.3928
4	-4855.44	6.0461	1	0.014	4.0e+07	20.3407	20.3613	20.3931
5	-4853.84	3.1904	1	0.074	4.0e+07	20.3383	20.3623	20.3993
6	-4851.58	4.5304	1	0.033	4.0e+07	20.333	20.3604	20.4027
7	-4847.61	7.942	1	0.005	3.9e+07	20.3205	20.3514	20.399
8	-4847.51	.20154	1	0.653	3.9e+07	20.3243	20.3586	20.4115
9	-4847.51	.00096	1	0.975	3.9e+07	20.3285	20.3662	20.4244
10	-4847.43	.16024	1	0.689	4.0e+07	20.3323	20.3735	20.437
11	-4831.38	32.094	1	0.000	3.7e+07	20.2694	20.3139	20.3828
12	-4818.46	25.834	1	0.000	3.5e+07	20.2195	20.2675	20.3416*
13	-4815.64	5.6341	1	0.018	3.5e+07	20.2119	20.2633	20.3427
14	-4813.98	3.321	1	0.068	3.5e+07	20.2091	20.264	20.3487
15	-4813.38	1.2007	1	0.273	3.5e+07	20.2108	20.2691	20.3591
16	-4810.57	5.6184	1	0.018	3.5e+07	20.2032	20.265	20.3603
17	-4808.7	3.7539	1	0.053	3.5e+07	20.1996	20.2647	20.3653
18	-4806.12	5.1557	1	0.023	3.4e+07	20.193	20.2616	20.3674
19	-4804.6	3.0319	1	0.082	3.4e+07	20.1908	20.2628	20.374
20	-4804.6	2.7e-05	1	0.996	3.5e+07	20.195	20.2704	20.3869
21	-4797.33	14.542	1	0.000	3.4e+07	20.1688	20.2476	20.3694
22	-4794.2	6.2571*	1	0.012	3.3e+07*	20.1598*	20.2422*	20.3692
23	-4793.42	1.5626	1	0.211	3.3e+07	20.1608	20.2465	20.3788
24	-4792.85	1.1533	1	0.283	3.3e+07	20.1625	20.2517	20.3893

```
| 25 | -4792.78  .13518    1  0.713  3.4e+07   20.1664    20.259    20.402  |
+-----------------------------------------------------------------------------+
 Endogenous:  D.sexpend
  Exogenous:  L.sexpend  _cons

. dfuller sexpend, lags(22)

Augmented Dickey-Fuller test for unit root         Number of obs   =       481

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -1.627            -3.442            -2.871            -2.570
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.4689

. dfuller sexpend, lags(12)

Augmented Dickey-Fuller test for unit root         Number of obs   =       491

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -2.399            -3.441            -2.870            -2.570
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.1420
```

$\Longrightarrow$ Unit root is detected.

8.4 Cointegration (共和分)

1. For a scalar y_t, when $\Delta y_t = y_t - y_{t-1}$ is a white noise (i.e., iid), we write $\Delta y_t \sim I(1)$.

2. **Definition of Cointegration:**

   Suppose that each series in a $g \times 1$ vector y_t is $I(1)$, i.e., each series has unit root, and that a linear combination of each series (i.e, $a' y_t$ for a nonzero vector a) is $I(0)$, i.e., stationary.

   Then, we say that y_t has a cointegration.

   a is called the cointegrating vector.

3. **Example:**

   Suppose that $y_t = (y_{1,t},\ y_{2,t})'$ is the following vector autoregressive process:

$$
\begin{aligned}
y_{1,t} &= \phi_1 y_{2,t} + \epsilon_{1,t}, \\
y_{2,t} &= y_{2,t-1} + \epsilon_{2,t}.
\end{aligned}
$$

   Then,

$$
\Delta y_{1,t} = \phi_1 \epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \quad \text{(MA(1) process)},
$$

$$\Delta y_{2,t} = \epsilon_{2,t},$$

where both $y_{1,t}$ and $y_{2,t}$ are $I(1)$ processes.

The linear combination $y_{1,t} - \phi_1 y_{2,t}$ is $I(0)$.

In this case, we say that $y_t = (y_{1,t},\ y_{2,t})'$ is cointegrated with $a = (1,\ -\phi_1)$.

$a = (1,\ -\phi_1)$ is called the cointegrating vector, which is not unique.

Therefore, the first element of a is set to be one.

8.5 Spurious Regression (見せかけ回帰)

1. Suppose that $y_t \sim I(1)$ and $x_t \sim I(1)$.

   For the regression model $y_t = x_t\beta + u_t$, OLS does not work well if we do not have the β which satisfies $u_t \sim I(0)$.

   $\Longrightarrow$ **Spurious regression (見せかけ回帰)**

2. Suppose that $y_t \sim I(1)$, y_t is a $g \times 1$ vector and $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$.

   $y_{2,t}$ is a $k \times 1$ vector, where $k = g - 1$.

Consider the following regression model:

$$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t, \qquad t = 1, 2, \cdots, T.$$

OLSE is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}.$$

Next, consider testing the null hypothesis $H_0: \ R\gamma = r$, where R is a $m \times k$ matrix ($m \leq k$) and r is a $m \times 1$ vector.

The F statistic, denoted by F_T, is given by:

$$F_T = \frac{1}{m}(R\hat{\gamma} - r)' \left(s_T^2 \begin{pmatrix} 0 & R \end{pmatrix} \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R' \end{pmatrix} \right)^{-1} (R\hat{\gamma} - r),$$

where

$$s_T^2 = \frac{1}{T-g} \sum_{t=1}^{T} (y_{1,t} - \hat{\alpha} - \hat{\gamma}' y_{2,t})^2.$$

When we have the γ such that $y_{1,t} - \gamma y_{2,t}$ is stationary, OLSE of γ, i.e., $\hat{\gamma}$, is not statistically equal to zero.

When the sample size T is large enough, H_0 is rejected by the F test.

3. Phillips, P.C.B. (1986) "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, Vol.33, pp.95 – 131.

Consider a $g \times 1$ vector y_t whose first difference is described by:

$$\Delta y_t = \Psi(L)\epsilon_t = \sum_{s=0}^{\infty} \Psi_s \epsilon_{t-s},$$

for ϵ_t an i.i.d. $g \times 1$ vector with mean zero , variance $\mathrm{E}(\epsilon_t \epsilon_t') = PP'$, and finite fourth moments and where $\{s\Psi_s\}_{s=0}^{\infty}$ is absolutely summable.

Let $k = g - 1$ and $\Lambda = \Psi(1)P$.

Partition y_t as $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$ and $\Lambda\Lambda'$ as $\Lambda\Lambda' = \begin{pmatrix} \Sigma_{11} & \Sigma_{21}' \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $y_{1,t}$ and Σ_{11} are scalars, $y_{2,t}$ and Σ_{21} are $k \times 1$ vectors, and Σ_{22} is a $k \times k$ matrix.

Suppose that $\Lambda\Lambda'$ is nonsingular,and define $\sigma_1^{*2} = \Sigma_{11} - \Sigma_{21}'\Sigma_{22}^{-1}\Sigma_{21}$.

Let L_{22} denote the Cholesky factor of Σ_{22}^{-1}, i.e., L_{22} is the lower triangular matrix satisfying $\Sigma_{22}^{-1} = L_{22}L_{22}'$.

Then, (a) – (c) hold.

(a) OLSEs of α and γ in the regression model $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$, denoted by $\hat{\alpha}_T$ and $\hat{\gamma}_T$, are characterized by:

$$\begin{pmatrix} T^{-1/2}\hat{\alpha}_T \\ \hat{\gamma}_T - \Sigma_{22}^{-1}\Sigma_{21} \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1^* h_1 \\ \sigma_1^* L_{22} h_2 \end{pmatrix},$$

where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & \int_0^1 W_2^*(r)'\mathrm{d}r \\ \int_0^1 W_2^*(r)\mathrm{d}r & \int_0^1 W_2^*(r)W_2^*(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} \int_0^1 W_1^*(r)\mathrm{d}r \\ \int_0^1 W_2^*(r)W_1^*(r)\mathrm{d}r \end{pmatrix}.$

$W_1^*(r)$ and $W_2^*(r)$ denote scalar and g-dimensional standard Brownian motions, and $W_1^*(r)$ is independent of $W_2^*(r)$.

(b) The sum of squared residuals, denoted by $\mathrm{RSS}_T = \sum_{t=1}^{T} \hat{u}_t^2$, satisfies

$$T^{-2}\mathrm{RSS}_T \longrightarrow \sigma_1^{*2} H,$$

where $H = \int_0^1 (W_1^*(r))^2\mathrm{d}r - \left(\begin{pmatrix} \int_0^1 W_1^*(r)\mathrm{d}r \\ \int_0^1 W_2^*(r)W_1^*(r)\mathrm{d}r \end{pmatrix}' \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right)^{-1}.$

(c) The F_T test satisfies:

$$T^{-1}F_T \longrightarrow \frac{1}{m}(\sigma_1^* R^* h_2 - r^*)' \times \left(\sigma_1^{*2} H \begin{pmatrix} 0 & R^* \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W_2^*(r)'\mathrm{d}r \\ \int_0^1 W_2^*(r)\mathrm{d}r & \int_0^1 W_2^*(r)W_2^*(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} 0 & R^* \end{pmatrix}' \right)^{-1}$$

$$\times(\sigma_1^* R^* h_2 - r^*),$$

where $R^* = RL_{22}$ and $r^* = r - R\Sigma_{22}^{-1}\Sigma_{21}$.

Summary: Spurious regression (見せかけの回帰)

Consider the regression model: $y_{1.t} = \alpha + y_{2,t}\gamma + u_t$ for $t = 1, 2, \cdots, T$

and $y_t \sim I(1)$ for $y_t = (y_{1,t}, y_{2,t})'$.

(a) indicates that OLSE $\hat{\gamma}_T$ is not consistent.

(b) indicates that $s_T^2 = \dfrac{1}{T-g}\sum_{t=1}^{T}\hat{u}_t^2$ diverges.

(c) indicates that F_T diverges.

$\Longrightarrow$ It seems that the coefficients are statistically significant, based on the conventional t statistics.