Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2}$$

6. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2)-\rho(1)^2}{1-\rho(1)^2} = 0$$

7. Estimation of AR(1) model:

   (a) Likelihood function

$$\log f(y_T, \cdots, y_1) = \log f(y_1) + \sum_{t=1}^{T} \log f(y_t|y_{t-1}, \cdots, y_1)$$

$$= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi_1^2}\right) - \frac{1}{\sigma^2/(1-\phi_1^2)}y_1^2$$

$$-\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right)$$

$$-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

Note as follows:

$$f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)$$

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of ϕ_1 and σ^2 satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\begin{aligned} \tilde{\phi}_1 &= \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2\tilde{\phi}_1}{1-\tilde{\phi}_1^2}\right)\Big/\sum_{t=2}^{T} y_{t-1}^2 \\ &\approx \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}, \qquad \text{when } T \text{ is large.} \end{aligned}$$

(b) Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^{T} (y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to ϕ_1.

$$\begin{aligned}
\hat{\phi}_1 &= \frac{\sum_{t=2}^{T} y_{t-1} y_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^{T} y_{t-1}\epsilon_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{(1/T)\sum_{t=2}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=2}^{T} y_{t-1}^2} \\
&\longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1
\end{aligned}$$

OLSE of ϕ_1 is a consistent estimator.

OLSE of ϕ_1 is equal to MLE when T is large.

The following equations are utilized.

$$\mathrm{E}(y_{t-1}\epsilon_t) = 0, \qquad \mathrm{E}(y_{t-1}^2) = \mathrm{Var}(y_{t-1}) = \gamma(0)$$

8. Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

Proof:

$y_{t-1}\epsilon_t$, $t = 1, 2, \cdots, T$, are distributed with mean zero and variance $\dfrac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T)\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1-\phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0, \frac{\sigma_\epsilon^4}{1-\phi_1^2}).$$

Next,

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1-\phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T})\sum_{t=1}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N(0, 1-\phi_1^2)$$

9. Some formulas:

   (a) Central Limit Theorem

   Random variables x_1, x_2, $\cdots$, x_T are mutually independently distributed with mean μ and variance σ^2.

   Define $\overline{x} = (1/T)\sum_{t=1}^{T} x_t$.

   Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} = \frac{\overline{x} - \mu}{\sigma/\sqrt{T}} \longrightarrow N(0, 1)$$

   (b) Central Limit Theorem II

   Random variables x_1, x_2, $\cdots$, x_T are distributed with mean μ and variance σ^2.

Define $\overline{x} = (1/T)\sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} \longrightarrow N(0, 1)$$

(c) Let x and y be random variables.

y converges in distribution to a distribution, and x converges in probability to a fixed value.

Then, xy converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \qquad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$

10. **AR(1) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

Mean:

Using the lag operator,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L$.

Multiply $\phi(L)^{-1}$ on both sides. Then, when $|\phi_1| < 1$, we have:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$\begin{aligned} \mathrm{E}(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) \\ &= \phi(1)^{-1}\mu = \frac{\mu}{1-\phi_1} \end{aligned}$$

Example: AR(2) Model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$.

1. The stationarity condition is: two solutions of x from $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$ are outside the unit circle.

2. Rewriting the AR(2) model,

$$(1 - \phi_1 L - \phi_2 L^2) y_t = \epsilon_t.$$

   Let $1/\alpha_1$ and $1/\alpha_2$ be the solutions of $\phi(x) = 0$.

   Then, the AR(2) model is written as:

$$(1 - \alpha_1 L)(1 - \alpha_2 L) y_t = \epsilon_t,$$

   which is rewritten as:

$$y_t = \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \epsilon_t$$