$$= \left(\frac{\alpha_1/(\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2/(\alpha_1 - \alpha_2)}{1 - \alpha_2 L} \right) \epsilon_t$$

3. **Mean of AR(2) Model:**

   When y_t is stationary, i.e., α_1 and α_2 are inside the unit circle,

$$\mu = \mathrm{E}(y_t) = \mathrm{E}(\phi(L)\epsilon_t) = 0$$

4. **Autocovariance Function of AR(2) Model:**

$$\begin{aligned}
\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\
&= \mathrm{E}\Big((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}\Big) \\
&= \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \phi_2 \mathrm{E}(y_{t-2} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) \\
&= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}
\end{aligned}$$

The initial condition is obtained by solving the following three equations:

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2,$$

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0).$$

Therefore, the initial conditions are given by:

$$\gamma(0) = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_\epsilon^2}{(1-\phi_2)^2-\phi_1^2},$$
$$\gamma(1) = \frac{\phi_1}{1-\phi_2}\gamma(0) = \left(\frac{\phi_1}{1-\phi_2}\right)\left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_\epsilon^2}{(1-\phi_2)^2-\phi_1^2}.$$

Given $\gamma(0)$ and $\gamma(1)$, we obtain $\gamma(\tau)$ as follows:

$$\gamma(\tau) = \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2), \qquad \text{for } \tau = 2, 3, \cdots.$$

5. **Another solution for $\gamma(0)$:**

From $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2$,

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1\rho(1) - \phi_2\rho(2)}$$

where

$$\rho(1) = \frac{\phi_1}{1-\phi_2}, \qquad \rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2 + (1-\phi_2)\phi_2}{1-\phi_2}.$$

6. **Autocorrelation Function of AR(2) Model:**

Given $\rho(1)$ and $\rho(2)$,

$$\rho(\tau) = \phi_1\rho(\tau-1) + \phi_2\rho(\tau-2), \qquad \text{for } \tau = 3, 4, \cdots,$$

7. **$\boldsymbol{\phi_{k,k}}$ = Partial Autocorrelation Coefficient of AR(2) Process:**

$$
\begin{pmatrix}
1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\
\rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\
\vdots & \vdots & & \vdots & \vdots \\
\rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1
\end{pmatrix}
\begin{pmatrix}
\phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k}
\end{pmatrix}
=
\begin{pmatrix}
\rho(1) \\ \rho(2) \\ \vdots \\ \rho(k)
\end{pmatrix},
$$

for $k = 1, 2, \cdots$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Autocovariance Functions:

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0),$$

$$\gamma(\tau) = \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2), \qquad \text{for } \tau = 3, 4, \cdots.$$

Autocorrelation Functions:

$$\rho(1) = \phi_1 + \phi_2\rho(1) = \frac{\phi_1}{1-\phi_2},$$

$$\rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2,$$

$$\rho(\tau) = \phi_1\rho(\tau-1) + \phi_2\rho(\tau-2), \qquad \text{for } \tau = 3, 4, \cdots.$$

$$\phi_{1,1} = \rho(1) = \frac{\phi_1}{1 - \phi_2}$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2$$

$$\phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}$$

$$= \frac{(\rho(3)-\rho(1)\rho(2))-\rho(1)^2(\rho(3)-\rho(1))+\rho(2)\rho(1)(\rho(2)-1)}{(1-\rho(1)^2)-\rho(1)^2(1-\rho(2))+\rho(2)(\rho(1)^2-\rho(2))} = 0.$$

8. **Log-Likelihood Function — Innovation Form:**

$$\log f(y_T,\cdots,y_1) = \log f(y_2,y_1) + \sum_{t=3}^{T} \log f(y_t|y_{t-1},\cdots,y_1)$$

where

$$f(y_2,y_1) = \frac{1}{2\pi}\begin{vmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{vmatrix}^{-1/2} \exp\left(-\frac{1}{2}(y_1\ y_2)\begin{pmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{pmatrix}^{-1}\begin{pmatrix}y_1\\ y_2\end{pmatrix}\right),$$

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_t-\phi_1 y_{t-1}-\phi_2 y_{t-2})^2\right).$$

Note as follows:

$$\begin{pmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{pmatrix} = \gamma(0)\begin{pmatrix}1 & \rho(1)\\ \rho(1) & 1\end{pmatrix} = \gamma(0)\begin{pmatrix}1 & \phi_1/(1-\phi_2)\\ \phi_1/(1-\phi_2) & 1\end{pmatrix}.$$

9. **AR(2) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$

Mean:

Rewriting the AR(2)+drift model,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$.

Under the stationarity assumption, we can rewrite the AR(2)+drift model as follows:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Therefore,

$$\mathrm{E}(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) = \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1 - \phi_2}$$

Example: AR(p) model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$.

1. **Variance of AR(p) Process:**

   Under the stationarity condition (i.e., the p solutions of x from $\phi(x) = 0$ are outside the unit circle),

   $$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1 \rho(1) - \cdots - \phi_p \rho(p)}.$$

   Note that $\gamma(\tau) = \rho(\tau)\gamma(0)$.

   Solve the following simultaneous equations for $\tau = 0, 1, \cdots, p$:

   $$\begin{aligned}\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\ &= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \cdots + \phi_p \gamma(\tau - p), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \cdots + \phi_p \gamma(\tau - p) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}\end{aligned}$$

2. **Estimation of AR(p) Model:**

1. OLS:

$$\min_{\phi_1,\cdots,\phi_p} \sum_{t=p+1}^{T} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p})^2$$

2. MLE:

$$\max_{\phi_1,\cdots,\phi_p} \log f(y_T, \cdots, y_1)$$

where

$$\log f(y_T, \cdots, y_1) = \log f(y_p, \cdots, y_2, y_1) + \sum_{t=p+1}^{T} \log f(y_t | y_{t-1}, \cdots, y_1),$$

$$f(y_p, \cdots, y_2, y_1) = (2\pi)^{-p/2} |V|^{-1/2} \exp\left(-\frac{1}{2} (y_1 \ y_2 \ \cdots \ y_p) V^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \right)$$

$$V = \gamma(0)\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(p-2) & \rho(p-1) \\ \rho(1) & 1 & & \rho(p-3) & \rho(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(1) & 1 \end{pmatrix}$$

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}}\exp\Big(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \ \cdots \ - \phi_p y_{t-p})^2\Big)$$

3. Yule=Walker (ユール・ウォーカー) Equation:

Multiply $y_{t-1}, y_{t-2}, \cdots, y_{t-p}$ on both sides of $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t = y_t$, take expectations for each case, and divide by the sample variance $\hat{\gamma}(0)$.

$$\begin{pmatrix} 1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-2) & \hat{\rho}(p-1) \\ \hat{\rho}(1) & 1 & & \hat{\rho}(p-3) & \hat{\rho}(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & \hat{\rho}(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p) \end{pmatrix}$$

where

$$\hat{\gamma}(\tau) = \frac{1}{T}\sum_{t=\tau+1}^{T}(y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu}), \qquad \hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}.$$

3. **AR(p) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots \phi_p y_{t-p} + \epsilon_t$

Mean:

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$.

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t$$

Taking the expectation on both sides,

$$\begin{aligned} \mathrm{E}(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) = \phi(1)^{-1}\mu \\ &= \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p} \end{aligned}$$

4. **Partial Autocorrelation of AR(p) Process:**

$\phi_{k,k} = 0$ for $k = p + 1, p + 2, \cdots$.

6.4 MA Model

MA (Moving Average，移動平均) Model:

1. MA(q)

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q},$$

   which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

   where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

2. **Invertibility (反転可能性):**

   The q solutions of x from $\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q = 0$ are outside the unit circle.

   $\Longrightarrow$ MA(q) model is rewritten as AR(∞) model.

Example: MA(1) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1}$

1. **Mean of MA(1) Process:**

$$\mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1}) = \mathrm{E}(\epsilon_t) + \theta_1 \mathrm{E}(\epsilon_{t-1}) = 0$$

2. **Autocovariance Function of MA(1) Process:**

$$\gamma(0) = \mathrm{E}(y_t^2) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = \mathrm{E}(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2)$$

$$= \mathrm{E}(\epsilon_t^2) + 2\theta_1 \mathrm{E}(\epsilon_t \epsilon_{t-1}) + \theta_1^2 \mathrm{E}(\epsilon_{t-1}^2) = (1 + \theta_1^2)\sigma_\epsilon^2$$