1.

[1]

$$E\left[\frac{\partial logL(\theta;x)}{\partial\theta}\right]=\int\frac{\partial logL(\theta;x)}{\partial\theta}L(\theta;x)dx$$

$$=\int\frac{\frac{\partial L(\theta;x)}{\partial\theta}}{L(\theta;x)}L(\theta;x)dx$$

$$=\int\frac{\partial L(\theta;x)}{\partial\theta}dx$$

$$=\frac{\partial}{\partial\theta}\int L(\theta;x)dx$$

また

$$\int L(\theta;x)dx=1$$

$$\frac{\partial}{\partial\theta}\int L(\theta;x)dx=0$$

したがって

$$E\left[\frac{\partial logL(\theta;x)}{\partial\theta}\right]=0$$

確率変数 X に置き換えて

$$E\left[\frac{\partial logL(\theta;X)}{\partial\theta}\right]=0$$

[2]

$$E\left[\frac{\partial logL(\theta;x)}{\partial\theta}\right]=\int\frac{\partial\, log\, L(\theta;x)}{\partial\theta}L(\theta;x)dx=0$$

θに対して微分して

$$\int\frac{\partial^2 logL(\theta;x)}{\partial\theta\partial\theta'}L(\theta;x)dx+\int\frac{\partial logL(\theta;x)}{\partial\theta}\frac{\partial L(\theta;x)}{\partial\theta'}dx=0$$

$$\int\frac{\partial^2 logL(\theta;x)}{\partial\theta\partial\theta'}L(\theta;x)dx+\int\frac{\partial logL(\theta;x)}{\partial\theta}\frac{\partial logL(\theta;x)}{\partial\theta'}L(\theta;x)dx=0$$

$$E\left(\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}\right)+E\left(\frac{\partial logL(\theta;X)}{\partial\theta}\frac{\partial logL(\theta;X)}{\partial\theta'}\right)=0$$

$$-E\left(\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}\right)=E\left(\frac{\partial logL(\theta;X)}{\partial\theta}\frac{\partial logL(\theta;X)}{\partial\theta'}\right)=V\left(\frac{\partial logL(\theta;X)}{\partial\theta}\right)$$

したがって

$$I(\theta)=E\left[\left(\frac{\partial logL(\theta;X)}{\partial\theta}\right)^2\right]=V\left(\frac{\partial logL(\theta;X)}{\partial\theta}\right)$$

[3]

$$E\left(\hat{\theta}(X)\right)=\int\hat{\theta}(x)L(\theta;x)dx$$

θに対して微分して

$$\frac{\partial E(\hat{\theta}(X))}{\partial\theta}=\int\hat{\theta}(x)\frac{\partial L(\theta;x)}{\partial\theta}dx$$

$$=\int\hat{\theta}(x)\frac{\partial logL(\theta;x)}{\partial\theta}L(\theta;x)dx$$

$$=Cov(\hat{\theta}(X),\frac{\partial logL(\theta;X)}{\partial\theta})$$

$$\left(\frac{\partial E(\hat{\theta}(X))}{\partial\theta}\right)^2=\left(Cov(\hat{\theta}(X),\frac{\partial logL(\theta;X)}{\partial\theta})\right)^2$$

$$=\rho^2V(\hat{\theta}(X))V(\frac{\partial logL(\theta;X)}{\partial\theta})$$

ここで

$$\rho=\frac{Cov(\hat{\theta}(X),\frac{\partial logL(\theta;X)}{\partial\theta})}{\sqrt{V(\hat{\theta}(X))}\sqrt{V(\frac{\partial logL(\theta;X)}{\partial\theta})}}$$

$$|\rho|\leq 1$$

$$\left(\frac{\partial E(\hat{\theta}(X))}{\partial\theta}\right)^2\leq V(\hat{\theta}(X))V(\frac{\partial logL(\theta;X)}{\partial\theta})$$

$$V(\hat{\theta}(X))\geq\frac{\left(\frac{\partial E(\hat{\theta}(X))}{\partial\theta}\right)^2}{V(\frac{\partial logL(\theta;X)}{\partial\theta})}$$

$\hat{\theta}(X)$はθの不偏推定量である

$$E\left(\hat{\theta}(X)\right)=\theta$$

$$V\left(\hat{\theta}(X)\right)\geq\frac{1}{-E\left(\frac{\partial^2 logL(\theta;X)}{\partial\theta^2}\right)}=I(\theta)^{-1}$$

$$V\left(\hat{\theta}(X)\right)\geq I(\theta)^{-1}$$

[4]
中心極限定理より

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{\partial logf(X_i;\theta)}{\partial\theta}-E(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial logf(X_i;\theta)}{\partial\theta})}{\sqrt{V(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial logf(X_i;\theta)}{\partial\theta})}}$$

$$=\frac{\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta}-E(\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta})}{\sqrt{V(\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta})}}$$

また

$$E\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial logf(X_i;\theta)}{\partial\theta}\right)=E\left(\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta}\right)=0$$

$$V\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial logf(X_i;\theta)}{\partial\theta}\right)=V\left(\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta}\right)=\frac{1}{n^2}I(\theta)$$

により

$$\frac{\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta}-\frac{1}{n}E(\frac{\partial logL(\theta;X)}{\partial\theta})}{\frac{1}{\sqrt{n}}\sqrt{\frac{1}{n}V(\frac{\partial logL(\theta;X)}{\partial\theta})}}=\frac{\frac{1}{n}\frac{\partial logL(\theta;X)}{\partial\theta}}{\frac{1}{\sqrt{n}}\sqrt{\frac{1}{n}I(\theta)}}=\frac{\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}}{\sqrt{\frac{1}{n}I(\theta)}}$$

$$\frac{\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}}{\sqrt{\frac{1}{n}I(\theta)}}\longrightarrow N(0,1)$$

さらに

$$\sigma^2=-\lim_{n\to\infty}\frac{1}{n}E\left[\frac{\partial^2 logL(\theta;X)}{\partial\theta^2}\right]=\lim_{n\to\infty}\frac{1}{n}I(\theta)$$

したがって

$$\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}\longrightarrow N(0,\sigma^2)$$

[5]

θの最尤推定量を$\tilde{\theta}$とする

$$\frac{1}{\sqrt{n}}\frac{\partial logL(\tilde{\theta};X)}{\partial\theta}$$

$$0=\frac{1}{\sqrt{n}}\frac{\partial logL(\tilde{\theta};X)}{\partial\theta}\approx\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}+\frac{1}{\sqrt{n}}\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}(\tilde{\theta}-\theta)$$

$$-\frac{1}{\sqrt{n}}\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}(\tilde{\theta}-\theta)\approx\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}\longrightarrow N(0,\sigma^2)$$

$$-\frac{1}{\sqrt{n}}\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}(\tilde{\theta}-\theta)=\sqrt{n}\left(-\frac{1}{n}\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}\right)(\tilde{\theta}-\theta)$$

$$\sqrt{n}(\tilde{\theta}-\theta)\approx\left(-\frac{1}{n}\frac{\partial^2 logL(\theta;X)}{\partial\theta\partial\theta'}\right)^{-1}\left(\frac{1}{\sqrt{n}}\frac{\partial logL(\theta;X)}{\partial\theta}\right)$$

$$\longrightarrow N(0,(\sigma^2)^{-1}\sigma^2(\sigma^2)^{-1})=N(0,(\sigma^2)^{-1})$$

[6]

$$Y = \lambda X$$

単調変換より

$$h(y) = X = \frac{y}{\lambda}$$

Yの密度関数$g(y)$は

$$g(y) = |h'(y)|f(h(y))$$

$$= |\frac{1}{\lambda}|f(\frac{y}{\lambda})$$

$$= \left|\frac{1}{\lambda}\right| \exp\left(-\frac{Y}{\lambda}\right) \qquad \frac{y}{\lambda} > 0$$

となる。

[7]

まず、制約付きの尤度関数

$$l(\theta = 1) = \prod_{i=1}^{n} f(x_i\,;\theta) = \prod_{i=1}^{n} \theta exp\,(-\frac{x_i}{\theta})$$

$$= \exp\,(-\sum_{i=1}^{n} x_i)$$

制約なしの尤度関数

$$l(\theta \neq 1) = \prod_{i=1}^{n} f(x_i\,;\theta) = \prod_{i=1}^{n} \theta exp\,(-\frac{x_i}{\theta})$$

$$= \theta^{-n} exp\,(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)$$

制約なしの最尤推定量

$$\log\,[\theta^{-n} exp\,(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)] = -nlog\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial logL(\theta; X)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} = 0$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}$$

次に、尤度比検定統計量について

$$\frac{l(\theta = 1)}{l(\hat{\theta} = \bar{X})} = \frac{\exp\,(-\sum_{i=1}^{n} x_i)}{\hat{\theta}^{-n} exp\,(-\frac{1}{\hat{\theta}}\sum_{i=1}^{n} x_i)} = \hat{\theta}^{n} \exp\,((\frac{1}{\hat{\theta}} - 1)\sum_{i=1}^{n} x_i) = \bar{X}^{n} \exp\,((\frac{1}{\bar{X}} - 1)\sum_{i=1}^{n} x_i)$$

したがって、n が大きいとき、自由度 1 カイ二乗分布になる。

$$-2log\frac{l(\theta = 1)}{l(\hat{\theta} = \bar{X})} = -2nlog\bar{X} - 2n(1 - \bar{X}) \longrightarrow \chi^2(1)$$