Econometrics II TA Session #10

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5 Time Series Analysis

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5 Time Series Analysis

- Exmple: AR(2) Model

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1. AR(1) Model
2. AR(2) Model

1. Stationarity condition

- The AR(1) model is given by

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

- The stationarity condition is : the solution of

$$\phi(x) = 1 - \phi_1 x = 0,$$

i.e., $x = \frac{1}{\phi_1}$ is greater than 1 in absolute value, or equivalently, $|\phi_1| < 1$.

2. Rewriting the Model

- Rewriting the AR(1) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1 (\phi_1 y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^2 (\phi_1 y_{t-3} + \epsilon_{t-2}) + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&\vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1} \\
&= \phi_1^s y_{t-s} + \sum_{k=1}^{s} \phi_1^{k-1} \epsilon_{t-k+1}.
\end{aligned}
$$

2. Rewriting the Model

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^s y_{t-s} + \sum_{k=1}^{s} \phi_1^{k-1} \epsilon_{t-k+1}.
\end{aligned}
$$

- As s is large, ϕ_1^s approaches zero. $(\because |\phi_1| < 1)$

3. MA representation

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1} \\
&= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \cdots \\
&= \sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}.
\end{aligned}
$$

- This is the moving average (MA) representation of the AR model.

4. Mean of AR(1) process

- The mean of AR(1) process is

$$
\begin{aligned}
\mu &= \mathbb{E}(y_t) \\
&= \mathbb{E}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \cdots) \\
&= \mathbb{E}(\epsilon_t) + \phi_1 \mathbb{E}(\epsilon_{t-1}) + \phi_1^2 \mathbb{E}(\epsilon_{t-2}) + \cdots \\
&= 0.
\end{aligned}
$$

5. Variance of AR(1) process

- The variance of AR(1) process is

$$
\begin{aligned}
\gamma(0) &= V(y_t) \\
&= V(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \cdots) \\
&= V(\epsilon_t) + V(\phi_1 \epsilon_{t-1}) + V(\phi_1^2 \epsilon_{t-2}) + \cdots && (\because \text{independence}) \\
&= V(\epsilon_t) + \phi_1^2 V(\epsilon_{t-1}) + \phi_1^4 V(\epsilon_{t-2}) + \cdots \\
&= \sigma^2 + \phi_1^2 \sigma^2 + \phi_1^4 \sigma^2 + \cdots && (\because \epsilon_t \sim (0, \sigma^2),\ \forall t) \\
&= \sigma^2 (1 + \phi_1^2 + \phi_1^4 + \cdots) \\
&= \frac{\sigma^2}{1 - \phi_1^2}.
\end{aligned}
$$

6. Autocovariance and Autocorrelation functions of AR(1) process

- Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1}\epsilon_{t-\tau+1}.$$

- Therefore, the autovcovariance function of AR(1) process is:

$$\begin{aligned}
\gamma(\tau) &= \mathbb{E}[(y_t - \mu)(y_{t-\tau} - \mu)] \\
&= \mathbb{E}(y_t y_{t-\tau}) \qquad (\because \mu = 0) \\
&= \mathbb{E}[(\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1}\epsilon_{t-\tau+1}) y_{t-\tau}] \\
&= \phi_1^\tau \mathbb{E}(y_{t-\tau} y_{t-\tau}) + \mathbb{E}(\epsilon_t y_{t-\tau}) + \phi_1 \mathbb{E}(\epsilon_{t-1} y_{t-\tau}) + \phi_1^{\tau-1}\mathbb{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\
&= \phi_1^\tau \gamma(0) = \frac{\sigma^2 \phi_1^\tau}{1-\phi_1^2}.
\end{aligned}$$

6. Autocovariance and Autocorrelation functions of AR(1) process

- Note that since ϵ_t and $y_{t-\tau}$ are independent,

$$\mathbb{E}(\epsilon_t y_{t-\tau}) = \mathbb{E}(\epsilon_t)\mathbb{E}(y_{t-\tau}) = 0 \times 0.$$

- The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \frac{\frac{\sigma^2 \phi_1^\tau}{1-\phi_1^2}}{\frac{\sigma^2}{1-\phi_1^2}} = \phi_1^\tau.$$

7. Another derivation of $\gamma(\tau)$

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

- Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation

$$\underbrace{\mathbb{E}(y_t y_{t-\tau})}_{=\gamma(\tau)} = \phi_1 \underbrace{\mathbb{E}(y_{t-1} y_{t-\tau})}_{=\gamma(\tau-1)} + \mathbb{E}(\epsilon_t y_{t-\tau}).$$

- Then we have

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau-1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau-1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

7. Another derivation of $\gamma(\tau)$

- Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by

$$\begin{aligned}
\gamma(0) &= \phi_1 \gamma(-1) + \sigma^2 \\
&= \phi_1 \gamma(1) + \sigma^2 && (\because \gamma(1) = \gamma(-1)) \\
&= \phi_1 \phi_1 \gamma(0) + \sigma^2 && (\because \gamma(1) = \phi_1 \gamma(0)) \\
&= \phi_1^2 \gamma(0) + \sigma^2. \\
\iff \quad \gamma(0) &= \frac{\sigma^2}{1 - \phi_2.}
\end{aligned}$$

- Autocovariance function $\gamma(\tau)$ is

$$\gamma(\tau) = \phi_1 \gamma(\tau - 1) = \phi_1^2 \gamma(\tau - 2) = \cdots = \phi_1^\tau \gamma(0) = \frac{\phi_1^\tau \sigma^2}{1 - \phi_2.}$$

8. Partial autocorrelation function of AR(1) process

- The partial atocorrelation function is denoted by $\phi_{k,k}$.
- In case of $k = 1$,

$$\phi_{1,1} = \rho(1) = \phi_1.$$

- In case of $k = 2$,

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = 0.$$

8. Partial autocorrelation function of AR(1) process

- The numerator of $\phi_{2,2}$ is

$$\begin{aligned}\rho(2)-\rho(1)^2 &= \frac{\gamma(2)}{\gamma(0)}-\left(\frac{\gamma(1)}{\gamma(0)}\right)^2 \\ &= \frac{\phi_1^2\gamma(0)}{\gamma(0)}-\left(\frac{\phi_1\gamma(0)}{\gamma(0)}\right)^2 \\ &= \phi_1^2-\phi_1^2=0.\end{aligned}$$

- Note that AR(1) model assumes that y_t does not depend on y_{t-2} directly, but via y_{t-1}.
- Thus, the autocorrelation between y_t and y_{t-2} has to be 0 after removing the influence of y_{t-1}.

9. Estimation of AR(1) model

- The unconditional and conditional distribution is given by

$$f(y_1) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1-\phi_1^2}}} \exp\left\{ -\frac{y_1^2}{\frac{2\sigma^2}{1-\phi_1^2}} \right\}$$

$$f(y_t | y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y_t - \phi_1 y_{t-1})^2}{2\sigma^2} \right\}.$$

- Note that the parameter interest is σ^2 and ϕ_1.

9. Estimation of AR(1) model

(a) The likelihood function is

$$
\begin{aligned}
\log f(y_T,\cdots,y_1) = &-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi_1^2}\right) - \frac{y_1^2}{\frac{2\sigma^2}{1-\phi_1^2}} \\
&-\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t-\phi_1 y_{t-1})^2 \\
= &-\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right) \\
&-\frac{y_1^2}{\frac{2\sigma^2}{1-\phi_1^2}} - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t-\phi_1 y_{t-1})^2.
\end{aligned}
$$

9. Estimation of AR(1) model

- The F.O.C.s are

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{y_1^2}{\frac{2(\sigma^2)^2}{1-\phi_1}} + \frac{1}{2(\sigma^2)^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0,$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1 y_1^2}{\sigma^2} + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0.$$

9. Estimation of AR(1) model

- The MLE of σ^2 and ϕ_1 are

$$\tilde{\sigma}^2 = \frac{1}{T}\left[(1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right],$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \frac{\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2 \tilde{\phi}_1}{1-\phi_1^2}}{\sum_{t=2}^{T} y_{t-1}^2}.$$

9. Estimation of AR(1) model

(b) OLS method:

- The loss function is

$$S(\phi_1) = \sum_{t=2}^{T} (y_t - \phi_1 y_{t-1})^2.$$

- The F.O.C. w.r.t. ϕ_1 is

$$\frac{\partial S(\phi)}{\partial \phi} = -2 \sum_{t=2}^{T} (y_t - \phi_1 y_{t-1}) y_{t-1} = 0.$$

9. Estimation of AR(1) model

- Solving this, we obtain the OLS estimator of ϕ

$$
\begin{aligned}
\widehat{\phi}_1 &= \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} \\
&= \phi_1 + \frac{\sum_{t=2}^{T} \epsilon_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} \\
&= \phi_1 + \frac{\frac{1}{T}\sum_{t=2}^{T} \epsilon_t y_{t-1}}{\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2} \\
&\xrightarrow{\mathbb{P}} \phi_1 + \frac{\mathbb{E}(\epsilon_t y_{t-1})}{\mathbb{E}(y_{t-1}^2)} && (\because \text{Ergodicity}) \\
&= \phi_1. && (\Rightarrow \text{The OLSE is consistent.})
\end{aligned}
$$

10. Asymptotic distribution of OLSE $\widehat{\phi}_1$

- We will show that

$$\sqrt{T}(\widehat{\phi}_1 - \phi_1) \xrightarrow[T\to\infty]{d} \mathcal{N}_{\mathbb{R}}(0, 1-\phi_1^2).$$

Proof

- By the expression above, we have

$$\widehat{\phi}_1 - \phi_1 = \frac{\sum_{t=2}^{T} \epsilon_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}$$

$$\iff \quad \sqrt{T}(\widehat{\phi}_1 - \phi_1) = \left[\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2\right]^{-1} \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \epsilon_t y_{t-1}.$$

10. Asymptotic distribution of OLSE $\widehat{\phi}_1$

- Assume that $\epsilon_t \sim \mathcal{N}_{\mathbb{R}}(0, \sigma_\epsilon^2)$.
- We have already confirmed $\mathbb{E}(\epsilon_t y_{t-1}) = 0$.
- Note that if two random variables X and Y are independent,

$$V(XY) = V(X)V(Y) + (\mathbb{E}[X])^2 V(Y) + (\mathbb{E}[Y])^2 V(X).$$

- Now, since ϵ_t and y_{t-1} are independent, the variance of the product is

$$\begin{aligned} V(\epsilon_t y_{t-1}) &= V(\epsilon_t)V(y_{t-1}) + [\mathbb{E}(\epsilon_t)]^2 V(y_{t-1}) + [\mathbb{E}(y_{t-1})]^2 V(\epsilon_t) \\ &= \sigma_\epsilon^2 \cdot \frac{\sigma_\epsilon^2}{1-\phi_1^2} = \frac{(\sigma_\epsilon^2)^2}{1-\phi_1^2}. \end{aligned}$$

10. Asymptotic distribution of OLSE $\widehat{\phi}_1$

- Then, we have

$$\epsilon_t y_{t-1} \sim \mathcal{N}_{\mathbb{R}}\left(0,\ \frac{(\sigma_\epsilon^2)^2}{1-\phi_1^2}\right) \Rightarrow \frac{1}{T}\sum_{t=2}^{T} \epsilon_t y_{t-1} \sim \mathcal{N}_{\mathbb{R}}\left(0,\ \frac{(\sigma_\epsilon^2)^2}{1-\phi_1^2}/T\right)$$

- By the central limit theorem,

$$\frac{\frac{1}{T}\sum_{t=2}^{T} \epsilon_t y_{t-1} - 0}{\sqrt{\frac{(\sigma_\epsilon^2)^2}{1-\phi_1^2}/T}} \xrightarrow[T\to\infty]{d} \mathcal{N}_{\mathbb{R}}(0,1) \Rightarrow \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \epsilon_t y_{t-1} \xrightarrow[T\to\infty]{d} \mathcal{N}_{\mathbb{R}}\left(0,\ \frac{(\sigma_\epsilon^2)^2}{1-\phi_1^2}\right).$$

10. Asymptotic distribution of OLSE $\widehat{\phi}_1$

- Next, by the ergodicity,

$$\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2 \xrightarrow[T\to\infty]{\mathbb{P}} \mathbb{E}(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1-\phi_1^2}.$$

- Again, by the ergodic theorem (theorem 20.2 in Greene),

$$\left[\frac{1}{T}\sum_{t=2}^{T} y_{t-1}^2\right]^{-1} \xrightarrow[T\to\infty]{\mathbb{P}} \left[\frac{\sigma_{\epsilon t}^2}{1-\phi_1^2}\right]^{-1}$$

10. Asymptotic distribution of OLSE $\widehat{\phi}_1$

- By the Slutsky theorem,

$$\sqrt{T}(\widehat{\phi}_1 - \phi_1) \xrightarrow[T\to\infty]{d} \mathcal{N}_{\mathbb{R}}\left(0,\ 1-\phi_1^2\right),$$

where the asymptotic variance can be derived as follows:

$$V\left(\sqrt{T}(\widehat{\phi}_1 - \phi_1)\right) \xrightarrow[T\to\infty]{\mathbb{P}} \left[\frac{\sigma_{\epsilon_t}^2}{1-\phi_1^2}\right]^{-1} \frac{(\sigma_{\epsilon}^2)^2}{1-\phi_1^2} \left[\frac{\sigma_{\epsilon_t}^2}{1-\phi_1^2}\right]^{-1} = 1-\phi_1^2.$$

12. AR(1) +drift

- Consider the model:

$$y_t = \mu + \phi_1 y_{t-1} + \epsilon_t.$$

- Using the lag operator, we have

$$\begin{aligned} y_t &= \mu + \phi_1 L y_t + \epsilon_t \\ \Longleftrightarrow \quad (1 - \phi_1 L) y_t &= \mu + \epsilon_t \\ \Longleftrightarrow \quad \phi(L) y_t &= \mu + \epsilon_t, \end{aligned}$$

where $\phi(L) := 1 - \phi_1 L$.

12. AR(1) +drift

- Multiply $\phi(L)^{-1}$ on both sides, then when $|\phi_1| < 1$, we have

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

- Taking the expectation, we obtain the mean of y_t:

$$\begin{aligned}\mathbb{E}(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}\mathbb{E}(\epsilon_t) \\ &= \frac{\mu}{1-\phi_1}.\end{aligned}$$

1. AR(1) Model
2. AR(2) Model

1. Stationarity condition

- The stationarity condition is that 2 solutions of x from

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$$

  are outside the unit circle.

- Notice that ϕ_1 and ϕ_2 are used as parameters in this equation.

  $\Rightarrow$ The solutions are function of these parameters.

  $\Rightarrow$ The stationarity condition is imposed on the values of the parameters.

2. Rewriting AR(2) model

- An AR(2) model is given by

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t.$$

- Rewriting this expression,

$$\begin{aligned} y_t &= \phi_1 L y_t + \phi_2 L^2 y_t + \epsilon_t \\ \Longleftrightarrow \quad (1 - \phi_1 L - \phi_2 L^2) y_t &= \epsilon_t \\ \Longleftrightarrow \quad \phi(L) y_t &= \epsilon_t, \end{aligned}$$

where $\phi(L) := 1 - \phi_1 L - \phi_2 L^2$.

2. Rewriting AR(2) model

- Let $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ be the solution of $\phi(L) = 0$.
- We then have

$$(1 - \alpha_1 L)(1 - \alpha_2 L) y_t = \epsilon_t.$$

- Arranging this, we have

$$\begin{aligned} y_t &= \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \epsilon_t \\ &= \left(\frac{\alpha_1/(\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2/(\alpha_1 - \alpha_2)}{1 - \alpha_2 L} \right) \epsilon_t. \end{aligned}$$

3. Mean of AR(2) model

- Recall that

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \\ \iff \quad \phi(L) y_t &= \epsilon_t. \end{aligned}$$

- When y_t is stationary, i.e., α_1 and α_2 are within the unit circle,

$$\mu := \mathbb{E}(y_t) = \mathbb{E}\left(\frac{\epsilon_t}{\phi(L)}\right) = 0.$$

4. Autocovariance function of AR(2) model

- The autocovariance function is

$$\begin{aligned}
\gamma(\tau) &= \mathbb{E}[(y_t - \mu)(y_{t-\tau} - \mu)] \\
&= \mathbb{E}(y_t y_{t-\tau}) \\
&= \mathbb{E}[(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}] \\
&= \phi_1 \mathbb{E}(y_{t-1} y_{t-\tau}) + \phi_2 \mathbb{E}(y_{t-2} y_{t-\tau}) + \mathbb{E}(\epsilon_t y_{t-\tau}) \\
&= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) & \text{for } \tau = 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2 & \text{for } \tau \neq 0. \end{cases}
\end{aligned}$$

4. Autocovariance function of AR(2) model

$$\gamma(\tau) = \begin{cases} \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2) & \text{for } \tau = 0, \\ \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2) + \sigma_\epsilon^2 & \text{for } \tau \neq 0. \end{cases}$$

- To obtain the initial condition, we consider the cases of $\tau = 0, 1, 2$:

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2$$

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1)$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0)$$

4. Autocovariance function of AR(2) model

- We solve the system of equations to obtain:

$$\gamma(0) = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_\epsilon^2}{(1-\phi_2)^2-\phi_1^2},$$

$$\gamma(1) = \left(\frac{\phi_1}{1-\phi_2}\right)\gamma(0),$$

$$\gamma(2) = \phi_1\gamma(1)+\phi_2\gamma(0) = \frac{\phi_1^2+(1-\phi_2)\phi_2}{1-\phi_2}\gamma(0).$$

- Given these initial condition, we obtain $\gamma(\tau)$ as follows:

$$\gamma(\tau) = \phi_1\gamma(\tau-1)+\phi_2\gamma(\tau-2), \text{ for } \tau = 3,4,\cdots .$$

5. Another solution for $\gamma(0)$

- Rearranging the expression above,

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2$$

$$\gamma(0) = \phi_1\rho(1)\gamma(0) + \phi_2\rho(2)\gamma(0) + \sigma_\epsilon^2 \quad \left(\because \rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}\right)$$

$$\iff \quad \Big[1 - \phi_1\rho(1) - \phi_2\rho(2)\Big]\gamma(0) = \sigma_\epsilon^2$$

$$\iff \quad \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1\rho(1) - \phi_2\rho(2)}.$$

where

$$\rho(1) = \frac{\phi_1}{1-\phi_2} \quad \text{and} \quad \rho(2) = \frac{\phi_1^2 + (1-\phi_2)\phi_2}{1-\phi_2}.$$

6. Autocorrelation function of AR(2) model

- Recall that the autocovariance function is given by:

$$\gamma(\tau) = \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), \text{ for } \tau = 3, 4, \cdots .$$

- Multiplying $1/\gamma(0)$ on both sides, we obtain the autocorrelation function:

$$\frac{\gamma(\tau)}{\gamma(0)} = \phi_1 \frac{\gamma(\tau - 1)}{\gamma(0)} + \phi_2 \frac{\gamma(\tau - 2)}{\gamma(0)}$$
$$\iff \rho(\tau) = \phi_1 \rho(\tau - 1) + \phi_2 \rho(\tau - 2), \text{ for } \tau = 3, 4, \cdots .$$

7. Partial autocorrelation coeeficient of AR(2) process

- The partial autocorrelation coefficient $\phi_{k,k}$ is expressed as

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & \cdots & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(k-2) & \rho(k-3) & \cdots & 1 & \rho(1) \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k-1) \\ \rho(k) \end{pmatrix}$$

for $k = 1, 2, \cdots$.

- Using the Cramer's rule,

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & \cdots & \rho(k-3) & \rho(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(k-2) & \rho(k-3) & \cdots & 1 & \rho(k-1) \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & \cdots & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(k-2) & \rho(k-3) & \cdots & 1 & \rho(1) \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}.$$

- Let us look at $\phi_{k,k}$ in case of $k = 1, 2, 3$.

$$\phi_{1,1} = \rho(1) = \frac{\phi_1}{1 - \phi_2}.$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2.$$

$$\phi_{3,3} = 0.$$

- Note that since we are now considering an AR(2) model, y_t is not dependent on y_{t-3} directly, but via y_{t-1} and y_{t-2}.

8. Log-likelihood function - Innovation form

- The joint density is given by

$$\log f(y_T. \cdots, y_1) = \log f(y_2, y_1) + \sum_{t=3}^{T} \log f(y_t | y_{t-1}, \cdots, y_1),$$

where

$$f(y_2, y_1) = \frac{1}{\sqrt{2\pi}} \begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\},$$

$$f(y_t | y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left\{ -\frac{1}{2\sigma_\epsilon^2} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2 \right\}.$$

8. Log-likelihood function - Innovation form

$$
\begin{aligned}
\begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}^{-\frac{1}{2}} &= \left(\gamma(0)^2 - \gamma(1)^2\right)^{-\frac{1}{2}} \\
&= \left[\gamma(0)^2 - \left(\frac{\phi_1}{1-\phi_2}\right)^2 \gamma(0)^2\right]^{-\frac{1}{2}} \\
&= \left[1 - \left(\frac{\phi_1}{1-\phi_2}\right)^2\right]^{-\frac{1}{2}} \gamma(0)^{-1}.
\end{aligned}
$$

8. Log-likelihood function - Innovation form

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (y_1^2 + y_2^2)\gamma(0) + 2y_1y_2\gamma(1)$$

$$= \left[(y_1^2 + y_2^2) + 2y_1y_2\frac{\phi_1}{1-\phi_2} \right] \gamma(0).$$

- Then, we can rewrite $f(y_2, y_1)$:

$$f(y_2, y_1) = \frac{1}{\sqrt{2\pi}} \left[1 - \left(\frac{\phi_1}{1-\phi_2} \right)^2 \right]^{-\frac{1}{2}} \gamma(0)^{-1} \exp\left\{ -\frac{1}{2} \left[(y_1^2 + y_2^2) + 2y_1y_2\frac{\phi_1}{1-\phi_2} \right] \gamma(0) \right\}$$

8. Log-likelihood function - Innovation form

- The log-likelihood function is expressed as follows:

$$
\begin{aligned}
\log f(y_T, \cdots, y_1) &= \log f(y_2, y_1) + \sum_{t=3}^{T} \log f(y_t | y_{t-1}, \cdots, y_1) \\
&= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[1 - \left(\frac{\phi_1}{1-\phi_2}\right)^2\right] \\
&\quad - \log\gamma(0) - \frac{1}{2}\left[(y_1^2 + y_2^2) + 2y_1y_2\frac{\phi_1}{1-\phi_2}\right]\gamma(0) \\
&\quad - \frac{T-2}{2}\log(2\pi) - \frac{T-2}{2}\log(\sigma_\epsilon^2) \\
&\quad - \frac{1}{2\sigma_\epsilon^2}\sum_{t=3}^{T}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2 .
\end{aligned}
$$