Econometrics II TA Session #4

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November 10th, 2021

Contents

1.Limited Dependent Variable Model (制限従属変数モデル)

- Buying a Car

$$y_i = x_i\beta + u_i$$

y_i: $expenditure\ for\ a\ car$, x_i : income, price of the car…

- Working-hours of Wife

$$y_i^* = x_i\beta + u_i$$

y_i: $represents\ working - hours\ of\ wife,$
x_i : the number of children, age, education, income of husband ...

1.1 Truncated Regression Model

- $y_i = X_i\beta + u_i, \quad u_i \sim N(0, \sigma^2)$
$when\ y_i > a, where\ a\ is\ a\ constant, i = 1,2, \dots, n.$

- $Consider\ the\ case\ of\ y_i > a, (\ when\ y_i \leq a, y_i\ is\ not\ observed),$
$suppose: u_i \sim N(0, \sigma^2) \Longrightarrow \frac{u_i}{\sigma} \sim N(0,1)$

- $E(u_i | y_i > a) = E(u_i | X_i\beta + u_i > a) =?$
$E(y_i | y_i > a) = E(y_i | X_i\beta + u_i > a) = ?$
$the\ estimator\ of\ MLE\ =?$

Review: truncated normal distribution

- Probability density function of normal distribution

$$X \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

- Probability density function and cumulative distribution function of standard normal distribution

$$X \sim N(0,1)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) d\mathrm{z} = \int_{-\infty}^{x} \phi(z) dz$$

- *f(x) and F(x) are given by:*

$$X \sim N(0, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x)^2}{2\sigma^2}\right) = \frac{1}{\sigma}\phi(\frac{x}{\sigma}),$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z)^2}{2\sigma^2}\right) dz = \Phi(\frac{x}{\sigma})$$

- Definition of a truncated normal distribution

$$X \sim N(\mu, \sigma^2), \mathrm{X} > \mathrm{a}$$

$$f(x|x > a) = \frac{f(x)}{\int_a^\infty f(x)dx} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx}$$

$$= \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

- Mean of truncated normal distribution

$$E(X|X>a)=\int_a^\infty xf(x|x>a)dx$$

$$=\frac{\int_a^\infty x\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx}{\int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx}$$

$$=\frac{\sigma\phi\left(\frac{a-\mu}{\sigma}\right)+\mu(1-\Phi\left(\frac{a-\mu}{\sigma}\right))}{1-\Phi(\frac{a-\mu}{\sigma})}$$

$$=\frac{\sigma\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}+\mu$$

- Transformation of numerators and denominators

$$\int_a^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$z = \frac{x-\mu}{\sigma}, z > \frac{a-\mu}{\sigma}.$

$$= \int_{\frac{a-\mu}{\sigma}}^{\infty} (\sigma z + \mu)(2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2} z^2) dz$$

$$= \sigma \int_{\frac{a-\mu}{\sigma}}^{\infty} z (2\pi)^{-1/2} \exp(-\tfrac{1}{2} z^2) dz + \mu \int_{\frac{a-\mu}{\sigma}}^{\infty} (2\pi)^{-1/2} \exp(-\tfrac{1}{2} z^2) dz$$

$t = \frac{1}{2} z^2, t > \frac{1}{2} (\frac{x-\mu}{\sigma})^2.$

$$= \sigma \int_{\frac{1}{2}(\frac{a-\mu}{\sigma})^2}^{\infty} (2\pi)^{-1/2} \exp(-t)\, dt + \mu\left(1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right)$$

$$= \sigma\phi\left(\frac{a-\mu}{\sigma}\right) + \mu(1 - \Phi\left(\frac{a-\mu}{\sigma}\right))$$

$$\int_{a}^{\infty}(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx = \int_{\frac{x-\mu}{\sigma}}^{\infty}(2\pi)^{-1/2}\exp(-\frac{1}{2}z^2)\,dz$$

$$= 1 - \int_{-\infty}^{\frac{a-\mu}{\sigma}}(2\pi)^{-1/2}\exp\left(-\frac{1}{2}z^2\right)dz$$

$$= 1 - \Phi(\frac{a-\mu}{\sigma})$$

$$E(X|X > a) = \frac{\sigma\phi\left(\frac{a-\mu}{\sigma}\right)+\mu(1-\Phi\left(\frac{a-\mu}{\sigma}\right))}{1-\Phi(\frac{a-\mu}{\sigma})} = \frac{\sigma\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})} + \mu$$

- The conditional expectation of u_i

$$E(u_i|X_i\beta + u_i > a) = \int_{a-X_i\beta}^{\infty} u_i \frac{f(u_i)}{1-F(a-X_i\beta)} du_i$$

$$=\int_{a-X_i\beta}^{\infty} \frac{u_i}{\sigma} \frac{\phi(\frac{u_i}{\sigma})}{1-\Phi(\frac{a-x_i\beta}{\sigma})} du_i$$

$$= \frac{\sigma\phi(\frac{a-x_i\beta}{\sigma})}{1-\Phi(\frac{a-x_i\beta}{\sigma})}$$

- The conditional expectation of y_i

$$E(y_i|y_i > a) = E(X_i\beta + u_i|\, X_i\beta + u_i > a)$$

$$= X_i\beta + E(u_i|X_i\beta + u_i > a)$$

$$= X_i\beta + \frac{\sigma\phi(\frac{a-X_i\beta}{\sigma})}{1-\Phi(\frac{a-X_i\beta}{\sigma})}$$

- $The\ estimator\ of\ MLE$

$$L(\beta,\sigma^2) = \prod_{i=1}^{n} \frac{f(y_i - X_i\beta)}{1 - F(a - X_i\beta)} = \prod_{i=1}^{n} \frac{1}{\sigma} \frac{\phi(\frac{y_i - X_i\beta}{\sigma})}{1 - \Phi(\frac{a - X_i\beta}{\sigma})}$$

- When we use the OLS method, we get a biased estimator.

$$E[\beta_{ols}|y_i > a] = (\sum_{i=1}^{n} X_i X'_i)^{-1} \sum_{i=1}^{n} X_i E[y_i|y_i > a]$$

$$= (\sum_{i=1}^{n} X_i X'_i)^{-1} \sum_{i=1}^{n} X_i \left[X_i\beta + \sigma \frac{\phi(\frac{a-X_i\beta}{\sigma})}{1-\Phi(\frac{a-X_i\beta}{\sigma})} \right]$$

$$= \beta + \sigma(\sum_{i=1}^{n} X_i X'_i)^{-1} \sum_{i=1}^{n} X_i \frac{\phi(\frac{a-X_i\beta}{\sigma})}{1-\Phi(\frac{a-X_i\beta}{\sigma})}$$

1.2Censored Regression Model or Tobit Model

Unlike in truncated model, there is no truncation here. The feature that distinguishes the censored regression model from usual regression model is that the dependent variable is censored.

$$y_i^* = X_i\beta + u_i, \quad u_i \mid X_i \sim N(0, \sigma^2)$$

$$y_i = \begin{cases} y_i^* & if\ y_i^* \geq 0 \\ 0 & if\ y_i^* < 0 \end{cases}$$

$$
\begin{aligned}
P(y_i = 0|X_i) = P(y_i^* < 0|X_i) &= P(X_i\beta + u_i < 0|X_i) \\
&= P(u_i < -X_i\beta|X_i) \\
&= P(\frac{u_i}{\sigma} < -\frac{X_i\beta}{\sigma}|X_i) \\
&= \Phi(-\frac{X_i\beta}{\sigma}) \\
&= 1 - \Phi\left(\frac{X_i\beta}{\sigma}\right)
\end{aligned}
$$

$$
\begin{aligned}
P(y_i = X_i\beta + u_i|X_i) &= P(u_i = y_i - X_i\beta|X_i) \\
&= \frac{1}{\sigma}\phi(\frac{y_i - X_i\beta}{\sigma})
\end{aligned}
$$

- Maximize the log-likelihood function to calculate β and σ^2.

$$
logL_i = \mathbf{1}(y_i = 0)\log\left[1 - \Phi(\frac{X_i\beta}{\sigma})\right] + \mathbf{1}(y_i > 0)\log[\frac{1}{\sigma}\phi(\frac{y_i - X_i\beta}{\sigma})]
$$

2.Count Data Model (計数データモデル)

- Poisson distribution:

$$P(X = x) = f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad for\ x = 0,1,2,\ldots$$

- The expectation of X:

$$E(X) = \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} \lambda\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda\sum_{x'=0}^{\infty} \frac{e^{-\lambda}\lambda^{x'}}{x'!} = \lambda$$

Note: $\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{x!} = 1.$

- Poisson count data model

Let $y_i \in \{0,1,2,\ldots\}$ (discrete numbers) and $y_i \sim \text{Poi}(\lambda)$. Poisson count data model is represented as

$$E(y_i) = \lambda_i = \exp(X_i\beta)$$

Where $\lambda_i > 0$, it is better to avoid the specification: $\lambda = X_i\beta$.

The joint distribution is:

$$f(y_1, y_2, \cdots, y_n) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!} = L(\beta)$$

- The log-likelihood function is:

$$logL(\beta) = -\sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} y_i log\lambda_i - \sum_{i=1}^{n} log y_i!$$

$$= -\sum_{i=1}^{n} \exp(X_i\beta) + \sum_{i=1}^{n} y_i X_i \beta - \sum_{i=1}^{n} log y_i!$$

- The first-order condition is:

$$\frac{\partial logL(\beta)}{\partial\beta} = -\sum_{i=1}^{n} X'_i \exp(X_i\beta) + \sum_{i=1}^{n} X'_i y_i = 0$$

- By the newton-Raphson method

$$\beta^{(j+1)} = \beta^{(j)} - \left(\frac{\partial^2 logL(\beta^{(j)})}{\partial\beta\partial\beta'}\right)^{-1} \frac{\partial logL(\beta^{(j)})}{\partial\beta}$$

Finally, we get

$$\beta^{(j+1)} = \beta^{(j)} - \left(-\sum_{i=1}^{n} X'_i X_i \exp(X_i\beta^{(j)})\right)^{-1} \left(-\sum_{i=1}^{n} X'_i \exp\left(X_i\beta^{(j)}\right) + \sum_{i=1}^{n} X'_i y_i\right)$$

2.1 Zero-Inflated Poisson Count Data Model

- Dependent variable counts rare event and contains too much zeros.

We assume that the probability of $y_i = j$ is decomposed of two regimes, then we have $y_i = j$ and Regime 1(R1), $y_i = j$ and Regime2(R2).

$P(y_i = 0)$ and $P(y_i = j)$ separately as follows:

$$P(y_i = 0) = P(y_i = 0|R1)P(R1) + P(y_i = 0|R2)P(R2)$$

$$P(y_i = j) = P(y_i = j|R1)P(R1) + P(y_i = j|R2)P(R2)$$

Assumption:

- $P(y_i = 0|\mathrm{R1}) = 1$ and $\mathrm{P}(y_i = \mathrm{j}|\mathrm{R1}) = 0$ for $\mathrm{j} = 1,2 \ldots,$
- $\lambda_i = \exp(X_i\beta)$
- $P(R1) = F(z_i\alpha)$
- $P(R2) = 1 - F(z_i\alpha)$
- $P(y_i = j|\mathrm{R2}) = \frac{e^{-\lambda_i}{\lambda_i}^{y_i}}{y_i!}$

Based on assumptions, we get:

$$P(y_i = j) = P(R1)I_i + P(y_i = j|R_2)P(R_2)$$

$$P(y_i = j) = F(z_i\alpha)\, I_i + \frac{e^{-\lambda_i}{\lambda_i}^{y_i}}{y_i!}(1 - F(z_i\alpha))$$

- Maximize the log-likelihood function to calculate α and β.

$$logL(\alpha,\beta) = \sum_{i=1}^{n} logP(y_i = \mathrm{j}) = \sum_{i=1}^{n} log\left(F(z_i\alpha)\, I_i + \frac{e^{-\lambda_i}{\lambda_i}^{y_i}}{y_i!}(1 - F(z_i\alpha)) \right)$$

3.GLS-Review

- GLS Regression model

$$y = X\beta + u \qquad\qquad u \sim N(0, \sigma^2\Omega), \Omega\ is\ n\times n$$

$$\min_{\beta}(y - X\beta)'\Omega^{-1}(y - X\beta)$$

- GLS estimator of β is given by

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u$$

$$E(b) = \beta, V(b) = \sigma^2(X'\Omega^{-1}X)^{-1}$$

- We apply OLS to the regression model, we get the estimator of β.

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega)$$

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$$

$$E(\hat{\beta}) = \beta, \quad V(\hat{\beta}) = \sigma^2(X'X)^{-1}\, X'\Omega X(X'X)^{-1}$$

- The difference between two variance is:

$$V(\hat{\beta}) - V(b) = \sigma^2(X'X)^{-1}\, X'\Omega X(X'X)^{-1} - \sigma^2(X'\Omega^{-1}X)^{-1}$$

$$= \sigma^2\left((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\right)\Omega\left((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\right)'$$

$$= \sigma^2 A\Omega A' > 0$$

b is more efficient than $\hat{\beta}$.