Econometrics II TA Session #7

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Schedule for TA sessions

- There are 4 sessions left, including today's.
  - Dec 8 (Sakamoto)
  - Dec 22 (Kiku)
  - Jan 12 (Sakamoto)
  - Jan 26 (Sakamoto)
- Ms. Kiku is in charge of creating sample answers to the assignment 1 and 2.

Contents

1. 4.1 Method of Moments
2. 4.2 Generalized Method of Moments
3. Empirical Application of GMM

Method of Moments

- What is the method of moments?
- I present the procedure of it:
  1. We define the moment condition:

$$\mathbb{E}[f(X;\beta)] = 0,$$

  where X denotes variables and β parameters, and f is a function of X.

  2. Replacing the expectation by the sample mean, we derive the estimator:

$$\frac{1}{n}\sum_{i=1}^{n} f(X;\beta) = 0 \;\Rightarrow\; \hat{\beta} = g(X).$$

Method of Moments

- Let us apply the method of moments to the regression model as follows:

$$y_i = x_i\beta + u_i, \quad i = 1, \cdots, n,$$

where x_i and u_i are assumed to be stochastic and $x_i \in \mathbb{R}^k$.

- Firstly, we define the moment condition:

$$\mathbb{E}(x'u) = 0,$$

which is called as the orthogonality condition.

Method of Moments

- Note that to obtain the orthogonality condition, we need to assume the exogeneity assumption:

$$\mathbb{E}[u|x] = 0,$$

where x_i and u_i, $i \in \{1, \cdots, n\}$ are realizations generated from random variables x and u, respectively.

- Using the law of iterated expectation, we have the orthogonality condition.

$$\begin{aligned}\mathbb{E}[x'u] &= \mathbb{E}\{\mathbb{E}[x'u|x]\} \\ &= \mathbb{E}\{x'\mathbb{E}[u|x]\} = 0.\end{aligned}$$

Method of Moments

- Secondly, using the moment condition, we derive the estimator.
- That is, we deduce the following relationship by the moment condition.

$$\mathbb{E}(x'u) = 0 \ \Rightarrow \ \frac{1}{n}\sum_{i=1}^{n} x_i' u_i = 0.$$

- Replacing u_i by $y_i - x_i\beta$, we have

$$\frac{1}{n}\sum_{i=1}^{n} x_i'(y_i - x_i\beta) = 0.$$

Method of Moments

- We can solve as follows:

$$\beta_{MM} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i' x_i\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} x_i' y_i\right)$$
$$= (X'X)^{-1}X'y,$$

which is the same as OLSE and MLE and where

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \in \mathcal{M}_{n\times k}(\mathbb{R}), \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$$

Consistency

- The moment condition $\mathbb{E}(x'u) = 0$ is necessary for the MM estimator to have the **consistency.**

$$\beta_{MM} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i' x_i\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} x_i' y_i\right)$$
$$= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} x_i' x_i\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} x_i' u_i\right)$$
$$\xrightarrow{\mathbb{P}} \beta + \left(\mathbb{E}[x'x]\right)^{-1} \mathbb{E}[x'u] = \beta,$$

where we use the law of large numbers and the Slutsky theorem.

Instrumental Variables

- Suppose that $\mathbb{E}[u|x] \neq 0$, that is, the exogeneity assumption is not met.
- In this case, we do NOT have the condition $\mathbb{E}[x'u] = 0$.
- Then, we need to consider another moment condition.
- Suppose that we have found vector $z \in \mathbb{R}^k$ such that

$$\mathbb{E}[z'u] = 0,$$

  which is the moment condition.
- We call such a vector z instrumental variables.

Instrumental Variables

- Fortunately, since we can get the moment condition, let us derive the estimator.
- By the moment condition, we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} z_i' u_i = 0 \\
\Longleftrightarrow\ &\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta_{IV}) = 0 \\
\Longleftrightarrow\ &\beta_{IV} = \left(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' y_i\right) \\
&= (Z'X)^{-1}Z'y.
\end{aligned}
$$

Instrumental Variables

- Where $Z \in \mathcal{M}_{n\times k}(\mathbb{R})$ and the inverse matrix of $Z'X$ is assumed to exist.
- Moreover, we assume the existence of the following moments:

$$\frac{1}{n}\sum_{i=1}^{n} z_i' x_i \xrightarrow{\mathbb{P}} \mathbb{E}[z'x] =: M_{zx},$$
$$\frac{1}{n}\sum_{i=1}^{n} z_i' z_i \xrightarrow{\mathbb{P}} \mathbb{E}[z'z] =: M_{zz}.$$

- Of course, by the moment condition, we have

$$\frac{1}{n}\sum_{i=1}^{n} z_i' u_i \xrightarrow{\mathbb{P}} \mathbb{E}[z'u] = 0.$$

Instrumental Variables

- Then, let us confirm the consistency of the instrumental estimator.

$$
\begin{aligned}
\beta_{IV} &= \left(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' y_i\right) \\
&= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right) \\
&\xrightarrow{\mathbb{P}} \beta + \left(\mathbb{E}[z'x]\right)^{-1} \mathbb{E}[z'u] \\
&= \beta + (M_{zx})^{-1} \times 0 \\
&= \beta.
\end{aligned}
$$

Instrumental Variables

- We check the asymptotic normality of β_{IV}.
- Arranging the equation above, we have

$$\sqrt{n}(\beta_{IV} - \beta) = \left(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\right)^{-1} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right).$$

- By the central limit theorem,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right) \xrightarrow{d} \mathcal{N}_{\mathbb{R}^k}(0, \sigma^2 M_{zz}).$$

Instrumental Variables

- Note that

$$
\begin{aligned}
Var\left(\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right)\right) &= \frac{1}{n}\sum_{i=1}^{n} Var(z_i' u_i) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[z_i' u_i u_i' z_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\{\mathbb{E}[z_i' u_i u_i' z_i | z_i]\} \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\{z_i' \; \mathbb{E}[u_i u_i' | z_i] \; z_i\}
\end{aligned}
$$

Instrumental Variables

$$
\begin{aligned}
Var\left(\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right)\right) &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{z_i' \; \mathbb{E}[u_i u_i' | z_i] \; z_i\} \\
&= \sigma^2 \times \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[z_i' z_i] \\
&= \sigma^2 \mathbb{E}[z_i' z_i] \\
&= \sigma^2 M_{zz},
\end{aligned}
$$

where we assume that $u_i \sim (0, \sigma^2)$.

Instrumental Variables

- By the Slutsky theorem, we have

$$\sqrt{n}(\beta_{IV} - \beta) \xrightarrow{d} \mathcal{N}_{\mathbb{R}^k}(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}^{-1}).$$

- Practically, for large n we use the following distribution:

$$\beta_{IV} \sim \mathcal{N}_{\mathbb{R}^k}\left(\beta, s^2 (Z'X)^{-1} Z'Z (Z'X)^{-1}\right),$$

where $s^2 := \frac{1}{n-k}(y - x\beta_{IV})'(y - x\beta_{IV})$ is the estimator of σ^2.

IV → GMM

- So far, we have assumed z_i is $1 \times k$ vector.
- This implies that the matrix $Z'X$ is square, allowing for taking the inverse of it.
- On the other hand, if z is $1 \times r$ where $r > k$, we can NOT take the inverse of $Z'X$.
- This situation requires the necessity of using GMM.

Generalized Method of Moments

- Why "general"?
- GMM allows for
  - the use of instrumental variables whose dimension is greater than k;
  - the heteroschedasticity and auto correlation of the error term; and
  - non-linear estimation.

Generalized Method of Moments

- Suppose that the exogeneity assumption is not met, i.e., some explanatory variables are correlated with the error term.
- Suppose also that we have the following moment condition:

$$\mathbb{E}[z'u] = 0,$$

where $z \in \mathbb{R}^r$ with $r > k$.

Generalized Method of Moments

- By the moment condition, we have

$$\frac{1}{n}\sum_{i=1}^{n} z_i' u_i = 0.$$

- To deal with the fact that $Z'X$ is not square, we consider the following problem:

$$\begin{aligned}&\min_{\beta} \left(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\right)' W \left(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\right)\\ =&\min_{\beta} (y - X\beta)' Z W Z' (y - X\beta),\end{aligned}$$

where $W \in \mathcal{M}_{r\times r}(\mathbb{R})$ is a weight matrix.

Generalized Method of Moments

- The inverse matrix of the variance-covariance matrix of $Z'u$ is one the candidates of W.
- Suppose that $Var(u) = \sigma^2\Omega$.
- Then,

$$\begin{aligned} Var(Z'u) &= Z'Var(u)Z \\ &= Z'\sigma^2\Omega Z \\ &= \sigma^2 Z'\Omega Z \\ &= W^{-1} \end{aligned}$$

Generalized Method of Moments

- Then, the minimization problem becomes

$$\min_{\beta} (y - X\beta)' Z (Z'\Omega Z)^{-1} Z' (y - X\beta).$$

- The first order condition gives:

$$X'Z(Z'\Omega Z)^{-1}Z'y = X'Z(Z'\Omega Z)^{-1}Z'X\beta_{GMM}$$

$$\beta_{GMM} = \left[X'Z(Z'\Omega Z)^{-1}Z'X\right]^{-1} X'Z(Z'\Omega Z)^{-1}Z'y,$$

which is the GMM estimator.

Consistency of GMM

- Let us check the consistency of GMM estimator.
- Assume that there exist the following moments:

$$\frac{1}{n}X'Z = \frac{1}{n}\sum_{i=1}^{n} x_i' z_i \xrightarrow{\mathbb{P}} \mathbb{E}[x'z] =: M_{xz},$$

$$\frac{1}{n}Z'\Omega Z = \frac{1}{n}\sum_{i=1}^{n} z_i' \Omega_{ii} z_i \xrightarrow{\mathbb{P}} \mathbb{E}[z'\Omega_{ii} z] =: M_{z\Omega z},$$

where Ω_{ii} denotes i-th diagonal element of Ω.

Consistency of GMM

- By the law of large numbers,

$$
\begin{aligned}
\beta_{GMM} &= \left[X'Z(Z'\Omega Z)^{-1}Z'X\right]^{-1} X'Z(Z'\Omega Z)^{-1}Z'y \\
&= \beta + \left[\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\frac{1}{n}Z'X\right]^{-1}\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\frac{1}{n}Z'u \\
&\xrightarrow{\mathbb{P}} \beta + (M_{xz}M_{z\Omega z}^{-1}M_{xz}')^{-1}M_{xz}M_{z\Omega z}^{-1} \times 0 \\
&= \beta.
\end{aligned}
$$

- This concludes that the GMM estimator is consistent.

Variance of GMM

- The variance of the GMM estimator is

$$
\begin{aligned}
Var(\beta_{GMM}) &= Var\left(\beta + \left[X'Z\left(Z'\Omega Z\right)^{-1}Z'X\right]^{-1}X'Z\left(Z'\Omega Z\right)^{-1}Z'u\right) \\
&= Var(\beta) + Var(Au) \\
&= A\,Var(u)\,A' \\
&= A\sigma^2\Omega A',
\end{aligned}
$$

where $A := \left[X'Z\left(Z'\Omega Z\right)^{-1}Z'X\right]^{-1}X'Z\left(Z'\Omega Z\right)^{-1}Z'$.

Variance of GMM

$$
\begin{aligned}
& Var(\beta_{GMM}) \\
=& A\sigma^2\Omega A' \\
=& \sigma^2 \left[X'Z\left(Z'\Omega Z\right)^{-1}Z'X\right]^{-1} X'Z\left(Z'\Omega Z\right)^{-1}Z'\Omega Z\left(Z'\Omega Z\right)^{-1}Z'X\left[X'Z\left(Z'\Omega Z\right)^{-1}Z'X\right]^{-1} \\
=& \sigma^2 \left[X'Z\left(Z'\Omega Z\right)^{-1}Z'X\right]^{-1}.
\end{aligned}
$$

Asymptotic Normality of GMM

- We prove the asymptotic normality of β_{GMM}.
- Arranging the expression above, we obtain

$$\sqrt{n}(\beta_{GMM} - \beta) = \left[\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\frac{1}{n}Z'X\right]^{-1}\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\sqrt{n}\frac{1}{n}Z'u.$$

- By the central limit theorem,

$$\sqrt{n}\frac{1}{n}Z'u \xrightarrow{d} \mathcal{N}_{\mathbb{R}^k}(0, \sigma^2 M_{z\Omega z}).$$

Asymptotic Normality of GMM

- By the Slutsky theorem,

$$\sqrt{n}(\beta_{GMM} - \beta) = \left[\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\frac{1}{n}Z'X\right]^{-1}\frac{1}{n}X'Z\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\sqrt{n}\frac{1}{n}Z'u$$
$$\xrightarrow{d} \mathcal{N}_{\mathbb{R}^k}\left(0, \sigma^2(M_{xz}M_{z\Omega z}^{-1}M_{xz}')^{-1}\right).$$

- Practically, we use the following distribution:

$$\beta_{GMM} \sim \mathcal{N}_{\mathbb{R}^k}\left(\beta, s^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}\right),$$

where $s^2 = \frac{1}{n-k}(y - X\beta_{GMM})'\Omega^{-1}(y - X\beta_{GMM})$.

Data and Methodology

- data set : the 1980 US census
- We investigate the effect of the median dollar value of owner-occupied housing (`hsngval`) on the median monthly gross rent (`rent`):

$$\texttt{rent}_i = \beta_0 + \beta_1 \texttt{hsngval}_i + \beta_2 \texttt{pcturban}_i + u_i,$$

  where i indexes states, `pcturban` denotes the percentage of the population of living in urban areas.

- The null hypothesis is $H_0 : \beta_1 = 0$.

Endogeneity issue

- Random shocks that affect rental rates in a state probably also affect housing values.
  $\Rightarrow$ We treat `hsngval` as endogenous.
- So, we use the following variables as instruments:
  - family income
  - region of the country

$$\mathtt{rent}_i = \beta_0 + \beta_1 \mathtt{hsngval}_i + \beta_2 \mathtt{pcturban}_i + u_i,$$

$$\mathtt{hsngval}_i = \pi_0 + \pi_1 \mathtt{faminc}_i + \pi_2 \mathtt{2.region}_i + \pi_3 \mathtt{3.region} + \pi_4 \mathtt{4.region}_i + v_i.$$

Code in Stata

- When you use Stata, type as follows:
  - `ivreg gmm y x1 x2 (x1 = z1 z2), wmatrix(robust)`
  - `gmm (y - {xb:x1 x2} - {b0}), inst(x2 z1 z2)`
- Where
  - `y`: dependent var.
  - `x1`: endogenous var.
  - `x2`: exogenous var. (included exogenous var.)
  - `z1, z2`: instrumental var. (excluded exogenous var.)

Results

```
. ivregress gmm rent pcturban (hsngval = faminc i.region), wmatrix(robust)

Instrumental variables GMM regression             Number of obs   =         50
                                                  Wald chi2(2)    =     112.09
                                                  Prob > chi2     =     0.0000
                                                  R-squared       =     0.6616
GMM weight matrix: Robust                         Root MSE        =     20.358
```

rent	Coefficient	Robust std. err.	z	P>|z|	[95% conf. interval]	
hsngval	.0014643	.0004473	3.27	0.001	.0005877	.002341
pcturban	.7615482	.2895105	2.63	0.009	.1941181	1.328978
_cons	112.1227	10.80234	10.38	0.000	90.95052	133.2949

```
Instrumented: hsngval
 Instruments: pcturban faminc 2.region 3.region 4.region
```

```
. gmm (rent - {xb:hsngval pcturban} - {b0}), inst(pcturban faminc reg2-reg4)
Step 1
Iteration 0:   GMM criterion Q(b) =   56115.03
Iteration 1:   GMM criterion Q(b) =  110.91583
Iteration 2:   GMM criterion Q(b) =  110.91583

Step 2
Iteration 0:   GMM criterion Q(b) =   .2406087
Iteration 1:   GMM criterion Q(b) =  .13672801
Iteration 2:   GMM criterion Q(b) =  .13672801  (backed up)

GMM estimation

Number of parameters =   3
Number of moments    =   6
Initial weight matrix: Unadjusted                 Number of obs   =      50
GMM weight matrix:     Robust
```

	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
/xb_hsngval	.0014643	.0004473	3.27	0.001	.0005877	.002341
/xb_pcturban	.7615482	.2895105	2.63	0.009	.1941181	1.328978
/b0	112.1227	10.80234	10.38	0.000	90.95052	133.2949

```
Instruments for equation 1: pcturban faminc reg2 reg3 reg4 _cons
```

Reference

- stata.com "gmm - Generalized method of moments estimation."
- stata.com "ivregress - Single-equation instrumental-variables regression."