Kronecker Product — Review:

1. A: $n \times m$, B: $T \times k$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}, \text{ which is a } nT \times mk \text{ matrix.}$$

2. A: $n \times n$, B: $m \times m$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad |A \otimes B| = |A|^m |B|^n,$$

$$(A \otimes B)' = A' \otimes B', \qquad \mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B).$$

3. For A, B, C and D such that the products are defined,

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

End of Review

Going back to the previous slide, using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Apply OLS to the above regression model.

$$\begin{aligned}
\hat{\beta} &= \Big(((I_n \otimes D_T)X)'(I_n \otimes D_T)X\Big)^{-1}((I_n \otimes D_T)X)'(I_n \otimes D_T)y \\
&= \Big(X'(I_n \otimes D_T'D_T)X\Big)^{-1}X'(I_n \otimes D_T'D_T)y \\
&= \Big(X'(I_n \otimes D_T)X\Big)^{-1}X'(I_n \otimes D_T)y.
\end{aligned}$$

Note that the inverse matrix of D_T is not available, because the rank of D_T is $T-1$, not T (full rank).

The rank of a symmetric and idempotent matrix is equal to its trace.

The fixed effect v_i is estimated as:

$$\hat{v}_i = \overline{y}_i - \overline{X}_i \hat{\beta}.$$

Possibly, we can estimate the following regression:

$$\hat{v}_i = Z_i \alpha + \epsilon_i,$$

where it is assumed that the individual-specific effect depends on Z_i.

The estimator of σ_u^2 is given by:

$$\hat{\sigma}_u^2 = \frac{1}{nT - k - n} \sum_{i=1}^{n} \sum_{t=1}^{T} (y_{it} - X_{it}\hat{\beta} - \hat{v}_i)^2.$$

[Remark]

More than ten years ago, “fixed” indicates that v_i is nonstochastic.

Recently, however, “fixed” does not mean anything.

“fixed” indicates that OLS is applied and that v_i may be correlated with X_{it}.

Possibly, $\mathrm{E}(v_i|X) = \alpha_i(X)$, where $\alpha_i(X)$ is a function of X_{it} for $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$, and it is normalized to $\sum_{i=1}^{n} \alpha_i(X) = 0$.

3.2.2 Random Effect Model (ランダム効果モデル)

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where i indicates individual and t denotes time.

The assumptions on the error terms v_i and u_{it} are:

$$\mathrm{E}(v_i|X) = \mathrm{E}(u_{it}|X) = 0 \ \text{ for all } i,$$

$$\mathrm{V}(v_i|X) = \sigma_v^2 \ \text{ for all } i, \qquad \mathrm{V}(u_{it}|X) = \sigma_u^2 \ \text{ for all } i \text{ and } t,$$

$$\mathrm{Cov}(v_i, v_j|X) = 0 \ \text{ for } i \neq j, \qquad \mathrm{Cov}(u_{it}, u_{js}|X) = 0 \ \text{ for } i \neq j \text{ and } t \neq s,$$

$$\mathrm{Cov}(v_i, u_{jt}|X) = 0 \ \text{ for all } i, j \text{ and } t.$$

Note that X includes X_{it} for $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

In a matrix form with respect to $t = 1, 2, \cdots, T$, we have the following:

$$y_i = X_i\beta + v_i 1_T + u_i, \qquad i = 1, 2, \cdots, n,$$

where $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$, $X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{iT} \end{pmatrix}$ and $u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{pmatrix}$ are $T \times 1$, $T \times k$ and $T \times 1$, respectively.

$$u_i \sim N(0, \sigma_u^2 I_T) \;\text{ and }\; v_i 1_T \sim N(0, \sigma_v^2) \implies v_i 1_T + u_i \sim N(0, \sigma_v^2 1_T 1_T' + \sigma_u^2 I_T).$$

Again, in a matrix form with respect to i, we have the following:

$$y = X\beta + v + u,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, $v = \begin{pmatrix} v_1 1_T \\ v_2 1_T \\ \vdots \\ v_n 1_T \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ are $nT \times 1$, $nT \times k$, $nT \times 1$ and

$nT \times 1$, respectively.

The distribution of $u + v$ is given by:

$$v + u \sim N\Big(0, I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)$$

The likelihood function is given by:

$$L(\beta, \sigma_v^2, \sigma_u^2) = (2\pi)^{-nT/2} \Big| I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T) \Big|^{-1/2}$$
$$\times \exp\Big(-\frac{1}{2}(y - X\beta)'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1}(y - X\beta)\Big).$$

Remember that $f(x) = (2\pi)^{-k/2}|\Sigma|^{-1/2} \exp\Big(-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\Big)$ when $X \sim N(\mu, \Sigma)$, where X denotes a k-variate random variable.

The estimators of β, σ_v^2 and σ_u^2 are given by maximizing the following log-likelihood function:

$$
\begin{aligned}
\log L(\beta, \sigma_v^2, \sigma_u^2) = &-\frac{nT}{2}\log(2\pi) - \frac{1}{2}\log\Big|I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big| \\
&-\frac{1}{2}(y - X\beta)'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1}(y - X\beta).
\end{aligned}
$$

MLE of β, denoted by $\tilde{\beta}$, is given by:

$$
\begin{aligned}
\tilde{\beta} &= \Big(X'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1} X\Big)^{-1} X'\Big(I_n \otimes (\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)\Big)^{-1} y \\
&= \Big(\sum_{i=1}^{n} X_i'(\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)^{-1} X_i\Big)^{-1}\Big(\sum_{i=1}^{n} X_i'(\sigma_v^2 1_T 1_T' + \sigma_u^2 I_T)^{-1} y_i\Big),
\end{aligned}
$$

which is equivalent to GLS.

Note that $\tilde{\beta}$ is not operational, because $\hat{\beta}$ depends on σ_v^2 and σ_u^2.