3.3 Hausman's Specification Error (特定化誤差) Test

Regression model:

$$y = X\beta + u, \qquad y:\ n \times 1, \quad X:\ n \times k, \quad \beta:\ k \times 1, \quad u:\ n \times 1.$$

Suppose that X is stochastic.

If $\mathrm{E}(u|X) = 0$, OLSE $\hat{\beta}$ is unbiased because of $\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$ and $\mathrm{E}((X'X)^{-1}X'u) = 0$.

However, If $\mathrm{E}(u|X) \neq 0$, OLSE $\hat{\beta}$ is biased and inconsistent.

Therefore, we need to check if X is correlated with u or not.

$\Longrightarrow$ **Hausman's Specification Error Test**

The null and alternative hypotheses are:

- H_0 : X and u are independent, i.e., $\mathrm{Cov}(X, u) = 0$,

- H_1 : X and u are not independent.

Suppose that we have two estimators $\hat{\beta}_0$ and $\hat{\beta}_1$, which have the following properties:

- $\hat{\beta}_0$ is consistent and efficient under H_0, but is not consistent under H_1,

- $\hat{\beta}_1$ is consistent under both H_0 and H_1, but is not efficient under H_0.

Under the conditions above, we have the following test statistic:

$$(\hat{\beta}_1 - \hat{\beta}_0)'\Big(\mathrm{V}(\hat{\beta}_1) - \mathrm{V}(\hat{\beta}_0)\Big)^{-1}(\hat{\beta}_1 - \hat{\beta}_0) \longrightarrow \chi^2(k).$$

Example: $\hat{\beta}_0$ is OLS, while $\hat{\beta}_1$ is IV such as 2SLS.

Hausman, J.A. (1978) "Specification Tests in Econometrics," *Econometrica*, Vol.46, No.6, pp.1251–1271.

3.4 Choice of Fixed Effect Model or Random Effect Model

3.4.1 The Case where X is Correlated with u — Review

The standard regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I_n)$$

OLS is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u.$$

If X is not correlated with u, i.e., $\mathrm{E}(X'u) = 0$, we have the result: $\mathrm{E}(\hat{\beta}) = \beta$.

However, if X is correlated with u, i.e., $\mathrm{E}(X'u) \neq 0$, we have the result: $\mathrm{E}(\hat{\beta}) \neq \beta$.

$\Longrightarrow$ $\hat{\beta}$ is biased.

Assume that in the limit we have the followings:

$$
\begin{aligned}
(\frac{1}{n}X'X)^{-1} &\longrightarrow M_{xx}^{-1}, \\
\frac{1}{n}X'u &\longrightarrow M_{xu} \neq 0 \text{ when } X \text{ is correlated with } u.
\end{aligned}
$$

Therefore, even in the limit,

$$\operatorname{plim} \hat{\beta} = \beta + M_{xx}^{-1} M_{xu} \neq \beta,$$

which implies that $\hat{\beta}$ is not a consistent estimator of β.

Thus, in the case where X is correlated with u, OLSE $\hat{\beta}$ is neither unbiased nor consistent.

3.4.2 Fixed Effect Model or Random Effect Model

Usually, in the random effect model, we can consider that v_i is correlated with X_{it}.

[Reason:]

v_i includes the unobserved variables in the ith individual, i.e., ability, intelligence, and so on.

X_{it} represents the observed variables in the ith individual, i.e., income, assets, and so on.

The unobserved variables v_i are related to the observed variables X_{it}.

Therefore, we consider that v_i is correlated with X_{it}.

Thus, in the case of the random effect model, usually we cannot use OLS or GLS.

In order to use the random effect model, we need to test whether v_i is uncorrelated with X_{it}.

Apply Hausman's test.

- H_0 : X_{it} and e_{it} are independent ($\longrightarrow$ Use the random effect model),

- H_1 : X_{it} and e_{it} are not independent ($\longrightarrow$ Use the fixed effect model),

where $e_{it} = v_i + u_{it}$.

Note that:

- We can use the random effect model under H_0, but not under H_1.
- We can use the fixed effect model under both H_0 and H_1.
- The random effect model is more efficient than the fixed effect model under H_0.

Therefore, under H_0 we should use the random effect model, rather than the fixed effect model.

3.5 Applications

Example of Panel Data in Section 3: Production Function of Prefectures from 2001 to 2010.

pref: 都道府県（通し番号 1〜47）

year: 年度（2001〜2010 年）

y : 県内総生産（支出側、実質：固定基準年方式），出所：県民経済計算（平成 13 年度 - 平成 24 年度）（93SNA，平成 17 年基準計数）

k : 都道府県別民間資本ストック（平成 12 暦年価格，年度末，国民経済計算ベース　平成 23 年 3 月時点）一期前（2000〜2009 年）

l : 県内就業者数，出所：県民経済計算（平成 13 年度 - 平成 24 年度）（93SNA，平成 17 年基準計数）

```
. tsset pref year
```

```
       panel variable:  pref (strongly balanced)
        time variable:  year, 2001 to 2010
                delta:  1 unit

. gen ly=log(y)

. gen lk=log(k)

. gen ll=log(l)

. reg ly lk ll

      Source |       SS       df       MS              Number of obs =     470
-------------+------------------------------           F(  2,   467) =19374.95
       Model |  316.479302     2  158.239651           Prob > F      =  0.0000
    Residual |  3.81409572   467  .008167229           R-squared     =  0.9881
-------------+------------------------------           Adj R-squared =  0.9880
       Total |  320.293398   469  .682928354           Root MSE      =  .09037

------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          lk |   .0941587   .0081273    11.59   0.000     .0781881    .1101294
          ll |   .9976399   .0102641    97.20   0.000     .9774703    1.017809
       _cons |   .5970719   .0773137     7.72   0.000     .4451461    .7489978
------------------------------------------------------------------------------

. xtreg ly lk ll,fe
```

```
Fixed-effects (within) regression               Number of obs      =       470
Group variable: pref                            Number of groups   =        47

R-sq:  within  = 0.1721                         Obs per group: min =        10
       between = 0.9456                                        avg =      10.0
       overall = 0.9439                                        max =        10

                                                F(2,421)           =     43.77
corr(u_i, Xb)  = 0.8803                         Prob > F           =    0.0000

------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          lk |   .2329208   .0252321     9.23   0.000     .1833242    .2825175
          ll |   .3268537   .0810662     4.03   0.000     .1675088    .4861987
       _cons |   7.691145   1.376677     5.59   0.000     4.985128    10.39716
-------------+----------------------------------------------------------------
     sigma_u |  .41045507
     sigma_e |  .03561437
         rho |  .99252757   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(46, 421) =    56.22             Prob > F = 0.0000

. est store fixed

. xtreg ly lk ll,re

Random-effects GLS regression                   Number of obs      =       470
Group variable: pref                            Number of groups   =        47
```

```
R-sq:  within  = 0.1058                         Obs per group: min =         10
       between = 0.9805                                        avg =       10.0
       overall = 0.9787                                        max =         10

                                                Wald chi2(2)       =    3875.75
corr(u_i, X)   = 0 (assumed)                    Prob > chi2        =     0.0000

------------------------------------------------------------------------------
          ly |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          lk |   .2457767   .0153094    16.05   0.000     .2157708    .2757827
          ll |   .8105099   .0220256    36.80   0.000     .7673406    .8536793
       _cons |   .8332015   .2411141     3.46   0.001     .3606265    1.305776
-------------+----------------------------------------------------------------
     sigma_u |    .081609
     sigma_e |  .03561437
         rho |   .8400205   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. hausman fixed

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |     fixed          .          Difference          S.E.
-------------+----------------------------------------------------------------
          lk |    .2329208     .2457767       -.0128559         .020057
          ll |    .3268537     .8105099       -.4836562        .0780167
------------------------------------------------------------------------------
```

```
                  b = consistent under Ho and Ha; obtained from xtreg
   B = inconsistent under Ha, efficient under Ho; obtained from xtreg

Test:  Ho:  difference in coefficients not systematic

              chi2(2) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                      =      144.66
            Prob>chi2 =      0.0000
```

4 Generalized Method of Moments (GMM, 一般化積率法)

4.1 Method of Moments (MM, 積率法)

As $n \longrightarrow \infty$, we have the result: $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathrm{E}(X) = \mu$.

$\Longrightarrow$ **Law of Large Number (大数の法則)**

$X_1, X_2, \cdots, X_n$ are n realizations of X.

[Review] Chebyshev's inequality (チェビシェフの不等式) is given by:

$$P(|X-\mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \text{or} \quad P(|X-\mu| \leq \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2},$$

where $\mu = \mathrm{E}(X)$, $\sigma^2 = \mathrm{V}(X)$ and any $\epsilon > 0$.

Note that $P(|X-\mu| > \epsilon) + P(|X-\mu| \leq \epsilon) = 1$.

Replace X, $\mathrm{E}(X)$ and $\mathrm{V}(X)$ by $\overline{X}$, $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{V}(\overline{X}) = \dfrac{\sigma^2}{n}$.

As $n \longrightarrow \infty$,

$$P(|\overline{X} - \mu| \le \epsilon) \ge 1 - \frac{\sigma^2}{n\epsilon^2} \longrightarrow 1.$$

That is, $\overline{X} \longrightarrow \mu$ as $n \longrightarrow \infty$.

[End of Review]

$\overline{X}$ is an approximation of $\mathrm{E}(X) = \mu$.

Therefore, $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$ is taken as an estimator of μ.

$\Longrightarrow$ $\overline{X}$ is MM estimator of $\mathrm{E}(X) = \mu$.

MM is applied to the regression model as follows:

Regression model: $y_i = x_i\beta + u_i$, where x_i and u_i are assumed to be stochastic.

Familiar Assumption: $\mathrm{E}(x'u) = 0$, called the **orthogonality condition (直交条件)**, where x is a $1 \times k$ vector and u is a scalar.

We consider that $(x_1, x_2, \cdots, x_n)$ and $(u_1, u_2, \cdots, u_n)$ are realizations generated from random variables x and u, respectively.

From the law of large number, we have the following:

$$\frac{1}{n}\sum_{i=1}^{n} x_i' u_i = \frac{1}{n}\sum_{i=1}^{n} x_i'(y_i - x_i\beta) \longrightarrow \mathrm{E}(x'u) = 0.$$

Thus, the MM estimator of β, denoted by β_{MM}, satisfies:

$$\frac{1}{n}\sum_{i=1}^{n} x_i'(y_i - x_i\beta_{MM}) = 0.$$

Therefore, β_{MM} is given by:

$$\beta_{MM} = \Big(\frac{1}{n}\sum_{i=1}^{n} x_i' x_i\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i' y_i\Big) = (X'X)^{-1}X'y,$$

which is equivalent to OLS and MLE.

Note that X and y are:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$