- However, β_{MM} is inconsistent when $\mathrm{E}(x'u) \neq 0$, i.e.,

$$\beta_{MM} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u = \beta + \Big(\frac{1}{n}X'X\Big)^{-1}\Big(\frac{1}{n}X'u\Big) \not\longrightarrow \beta.$$

Note as follows:

$$\frac{1}{n}X'u = \frac{1}{n}\sum_{i=1}^{n} x_i'u_i \;\longrightarrow\; \mathrm{E}(x'u) \neq 0.$$

In order to obtain a consistent estimator of β, we find the instrumental variable z which satisfies $\mathrm{E}(z'u) = 0$.

Let z_i be the ith realization of z, where z_i is a $1 \times k$ vector.

Then, we have the following:

$$\frac{1}{n}Z'u = \frac{1}{n}\sum_{i=1}^{n} z_i'u_i \;\longrightarrow\; \mathrm{E}(z'u) = 0.$$

The β which satisfies $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} z_i'u_i = 0$ is denoted by β_{IV}, i.e., $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} z_i'(y_i - x_i\beta_{IV}) = 0$.

Thus, β_{IV} is obtained as:

$$\beta_{IV} = \Big(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} z_i' y_i\Big) = (Z'X)^{-1}Z'y.$$

Note that $Z'X$ is a $k \times k$ square matrix, where we assume that the inverse matrix of $Z'X$ exists.

Assume that as n goes to infinity there exist the following moment matrices:

$$\frac{1}{n}\sum_{i=1}^{n} z_i' x_i = \frac{1}{n} Z'X \longrightarrow M_{zx},$$
$$\frac{1}{n}\sum_{i=1}^{n} z_i' z_i = \frac{1}{n} Z'Z \longrightarrow M_{zz},$$
$$\frac{1}{n}\sum_{i=1}^{n} z_i' u_i = \frac{1}{n} Z'u \longrightarrow 0.$$

As n goes to infinity, β_{IV} is rewritten as:

$$\begin{aligned}\beta_{IV} &= (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u \\ &= \beta + (\frac{1}{n}Z'X)^{-1}(\frac{1}{n}Z'u) \longrightarrow \beta + M_{zx} \times 0 = \beta,\end{aligned}$$

Thus, β_{IV} is a consistent estimator of β.

- We consider the asymptotic distribution of β_{IV}.

By the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz})$$

Note that $\mathrm{V}(\frac{1}{\sqrt{n}}Z'u) = \frac{1}{n}\mathrm{V}(Z'u) = \frac{1}{n}\mathrm{E}(Z'uu'Z) = \frac{1}{n}\mathrm{E}\Big(\mathrm{E}(Z'uu'Z|Z)\Big)$

$= \frac{1}{n}\mathrm{E}\Big(Z'\mathrm{E}(uu'|Z)Z\Big) = \frac{1}{n}\mathrm{E}(\sigma^2 Z'Z) = \mathrm{E}(\sigma^2\frac{1}{n}Z'Z) \longrightarrow \mathrm{E}(\sigma^2 M_{zz}) = \sigma^2 M_{zz}.$

We obtain the following asymmptotic distribution:

$$\sqrt{n}(\beta_{IV} - \beta) = (\frac{1}{n}Z'X)^{-1}(\frac{1}{\sqrt{n}}Z'u) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}^{-1}{}')$$

Practically, for large n we use the following distribution:

$$\beta_{IV} \sim N\Big(\beta, s^2 (Z'X)^{-1} Z'Z (Z'X)^{-1}{}'\Big),$$

where $s^2 = \dfrac{1}{n-k}(y - X\beta_{IV})'(y - X\beta_{IV})$.

- In the case where z_i is a $1 \times r$ vector for $r > k$, $Z'X$ is a $r \times k$ matrix, which is not a square matrix. $\Longrightarrow$ **Generalized Method of Moments (GMM, 一般化積率法)**

4.2 Generalized Method of Moments (GMM, 一般化積率法)

In order to obtain a consistent estimator of β, we have to find the instrumental variable z which satisfies $\mathrm{E}(z'u) = 0$.

For now, however, suppose that we have z with $\mathrm{E}(z'u) = 0$.

Let z_i be the ith realization (i.e., the ith data) of z, where z_i is a $1 \times r$ vector and $r > k$.

Then, using the law of large number, we have the following:

$$\frac{1}{n}Z'u = \frac{1}{n}\sum_{i=1}^{n} z_i' u_i = \frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta) \longrightarrow \mathrm{E}(z'u) = 0.$$

The number of equations (i.e., r) is larger than the number of parameters (i.e., k).

Let us define W as a $r \times r$ weight matrix, which is symmetric.

We solve the following minimization problem:

$$\min_{\beta} \ \Big(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\Big)' W\Big(\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta)\Big),$$

which is equivalent to:

$$\min_{\beta} \ \Big(Z'(y - X\beta)\Big)' W\Big(Z'(y - X\beta)\Big),$$

i.e.,

$$\min_{\beta} \ (y - X\beta)'ZWZ'(y - X\beta).$$

Note that $\sum_{i=1}^{n} z_i'(y_i - x_i\beta) = Z'(y - X\beta)$.

W should be the inverse matrix of the variance-covariance matrix of $Z'(y-X\beta) = Z'u$.

Suppose that $\mathrm{V}(u) = \sigma^2\Omega$.

Then, $\mathrm{V}(Z'u) = \mathrm{E}(Z'u(Z'u)') = \mathrm{E}(Z'uu'Z) = Z'\mathrm{E}(uu')\mathrm{Z} = \sigma^2 Z'\Omega\mathrm{Z} = W^{-1}$.

The following minimization problem should be solved.

$$\min_{\beta} \; (y - X\beta)'\mathrm{Z}(\mathrm{Z}'\Omega\mathrm{Z})^{-1}Z'(y - X\beta).$$

The solution of β is given by the GMM estimator, denoted by β_{GMM}.

Remark: For the model: $y = X\beta + u$ and $u \sim (0, \sigma^2\Omega)$, solving the following minimization problem:

$$\min_{\beta}(y - X\beta)'\Omega^{-1}(y - X\beta),$$

GLS is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

Note that b is the best linear unbiased estimator.

Remark: The solution of the above minimization problem is equivalent to the GLE estimator of β in the following regression model:

$$Z'y = Z'X\beta + Z'u,$$

where Z, y, X, β and u are $n \times r$, $n \times 1$, $n \times k$, $k \times 1$ and $n \times 1$ matrices or vectors. Note that $r > k$.

$y^* = Z'y$, $X^* = Z'X$ and $u^* = Z'u$ denote $r \times 1$, $r \times k$ and $r \times 1$ matrices or vectors, where $r > k$.

Rewrite as follows:

$$y^* = X^*\beta + u^*,$$

$\Longrightarrow$ r is taken as the sample size.

u^* is a $r \times 1$ vector.

The elements of u^* are correlated with each other, beacuse each element of u^* is a function of $u_1, u_2, \cdots, u_n$.

The variance of u^* is:

$$\mathrm{V}(u^*) = \mathrm{V}(Z'u) = \sigma^2 Z'\Omega Z.$$

Go back to GMM:

$$\begin{aligned}
&(y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta) \\
&= y'Z(Z'\Omega Z)^{-1}Z'y - \beta'X'Z(Z'\Omega Z)^{-1}Z'y - y'Z(Z'\Omega Z)^{-1}Z'X\beta + \beta'X'Z(Z'\Omega Z)^{-1}Z'X\beta \\
&= y'ZWZ'y - 2y'Z(Z'\Omega Z)^{-1}Z'X\beta + \beta'X'Z(Z'\Omega Z)^{-1}Z'X\beta.
\end{aligned}$$

Note that $\beta'X'Z(Z'\Omega Z)^{-1}Z'y = y'Z(Z'\Omega Z)^{-1}Z'X\beta$ because both sides are scalars.

Remember that $\dfrac{\partial Ax}{x} = A'$ and $\dfrac{\partial x'Ax}{x} = (A + A')x$.

Then, we obtain the following derivation:

$$
\begin{aligned}
&\frac{\partial (y - X\beta)' Z(Z'\Omega Z)^{-1} Z'(y - X\beta)}{\partial \beta} \\
&\qquad = -2(y'Z(Z'\Omega Z)^{-1}Z'X)' + (X'Z(Z'\Omega Z)^{-1}Z'X + (X'Z(Z'\Omega Z)^{-1}Z'X)')\beta \\
&\qquad = -2X'Z(Z'\Omega Z)^{-1}Z'y + 2X'Z(Z'\Omega Z)^{-1}Z'X\beta = 0
\end{aligned}
$$

The solution of β is denoted by β_{GMM}, which is:

$$\beta_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'y.$$

The mean of β_{GMM} is asymptotically obtained.

$$
\begin{aligned}
\beta_{GMM} &= (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'(X\beta + u) \\
&= \beta + (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u \\
&= \beta + \Big((\frac{1}{n}X'Z)(\frac{1}{n}Z'\Omega Z)^{-1}(\frac{1}{n}Z'X)\Big)^{-1}(\frac{1}{n}X'Z)(\frac{1}{n}Z'\Omega Z)^{-1}(\frac{1}{n}Z'u)
\end{aligned}
$$